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Tame automorphism groups

w/ Stéphane Lamy

an automorphism of $\mathbb{R}^n \to \mathbb{R}^n$ is

a collection of n polynomials:

$(f_1, f_2, \ldots, f_n)$

where each $f_i = f_i(x_1, \ldots, x_m)$

is a polynomial in m variables.

further, we require that this n-tuple has an inverse of the same form.

Rk: can replace $\mathbb{R}$ with any other characteristic zero field.

Examples

(1) $(x_1, x_2) \longmapsto (\underbrace{x_1 + x_2}_{f_1}, \underbrace{x_1 - x_2}_{f_2})$

has inverse

$(x_1, x_2) \longmapsto \left(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}\right)$

(2) $(x_1, x_2) \longmapsto (x_1 + 10, x_2)$

(3) Any affine transformation

$A_n = GL_n \ltimes \mathbb{R}^n$

other examples (non-affine):

(4) $(x_1, x_2) \longmapsto (x_1, x_2 + x_1^2)$

has inverse:

$(x_1, x_2) \longmapsto (x_1, x_2 - x_1^2)$

(5) Elementary transformations

$E_2 = \{(x_1, x_2) \longmapsto (ax_1 + b, dx_2 + P(x_1))\}$

the elementary group.

Lemma: Any automorphism of the form

$(x_1, x_2) \xrightarrow{\varphi} (ax_1 + b, f(x_1, x_2))$

is an element $\varphi \in E_2$.

proof:

consider the differential

$$d\varphi = \begin{pmatrix} a & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix}$$

automorphism $\Rightarrow$ Jacobian = constant.

$J = a \cdot \frac{\partial f}{\partial x_2} - 0 = \text{constant}$

$\Rightarrow \frac{\partial f}{\partial x_2} = d$, so

$f(x_1, x_2) = d\, x_2 + P(x_1)$

$\square$

Jung (1942), van der Kulk (1953): the group Aut($\mathbb{R}^2$) is generated by A_2, E_2.

in particular,

$$\boxed{\mathrm{Aut}(\mathbb{R}^2) = A_2 *_{A_2 \cap E_2} E_2}$$

amalgamated product

∃

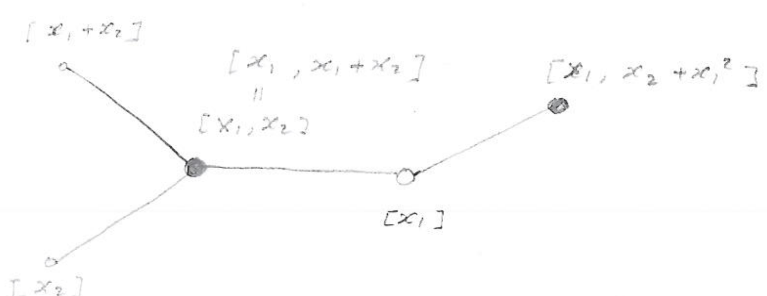
Let $T_{\mathbb{R}}$ = Bass-Serre tree.

$V(T_{\mathbb{R}})$ is partitioned into 2 types

- • cosets of A_2: automorphisms (f_1, f_2) modulo post composition with an affine map $[f_1, f_2]$
- ○ cosets of E_2: by Lemma, the f_1-component (1st component) needs to be fixed up to an affine map: $f_1 \mapsto af_1 + b$. (suffices to specify second component)

example tree.

$T_{\mathbb{R}}$ =



sp.?

Shestakov + Umirbaev (2004)

$$\mathrm{Aut}(\mathbb{R}^3) \neq \langle \underbrace{A_3}_{\text{affine}},\ (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + P(x_1, x_2)) \rangle$$

proof exhibits an automorphism not expressible in these words.

same

the TAME AUTOMORPHISM group of $\mathbb{R}^3$

$$\boxed{\mathrm{Tame}(\mathbb{R}^3) = \langle A_3,\ (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + P(x_1, x_2)) \rangle}$$

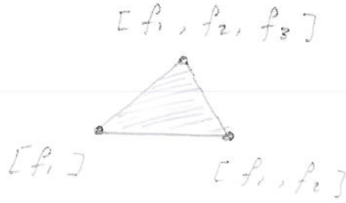
has a distinguished subgroup

Jacobian.

$$\boxed{\mathrm{STame}(\mathbb{R}^3) = \{\text{tame autom.'s with } J = 1\}}$$

def$^{\underline{n}}$ (Lamy): a type-k vertex ($k = 1, 2, 3$) is a k-tuple $(f_1, \dots, f_k)$ that can be extended to a tame autom$^{\underline{m}}$ $[f_1, f_2, f_3] \in \mathrm{Tame}(\mathbb{R}^3)$ modulo an affine map.

construct a complex $\mathcal{C}$, from triangles.



<u>Lamy (2015)</u>: $\mathcal{C}$ is simply connected

<u>Main Thm 1</u>: $\mathcal{C}$ is contractible + Gromov hyperbolic.

<u>Main Thm 2</u>: $\exists$ a loxodromic WPD element of $STame(\mathbb{R}^3) \curvearrowright \mathcal{C}$

* acylindrically hyperbolic *.

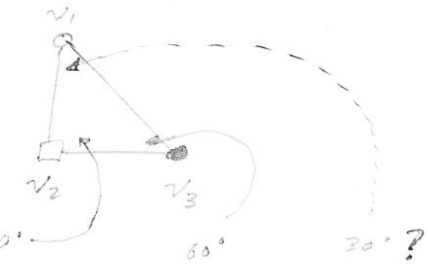
<u>Corollary</u>: work of Dahmani + Guirardel + Osin $\Rightarrow$ $STame(\mathbb{R}^3)$ is <u>NOT</u> simple.

proof

<u>Hurewicz</u>, <u>Whitehead</u> $\Rightarrow$ contractibility is implied by $S^2 \longrightarrow \mathcal{C}$ being homotopically trivial.

<u>idea</u>: attempt to equip image of S^2 with a NPC metric, (having a global NPC metric on $\mathcal{C}$ is too optimistic)

<u>Rk</u>: $\mathcal{C} / Tame(\mathbb{R}^3)$ =

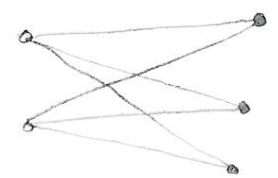


impose angles.

It suffices to understand the link of each vertex type

$lk(v_2)$ = complete bipartite graph

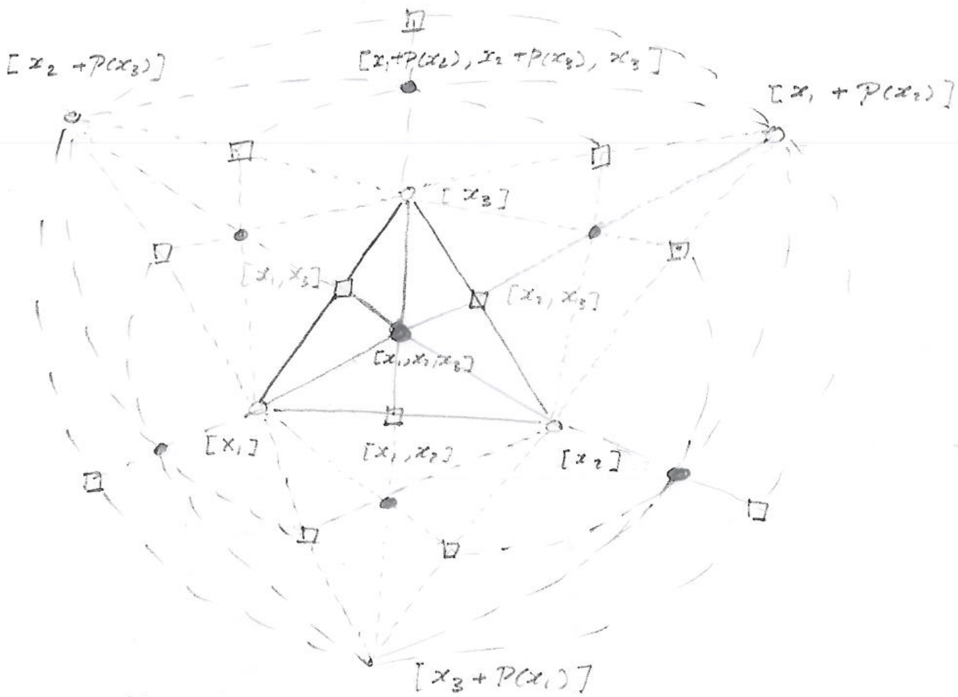
type-1 type-3



$lk(v_3)$ = incidence graph of the projective plane $\mathbb{R}P^2$.

$lk(v_1)$ is less clear.

Example (build local picture of $\mathcal{C}$)



Observations : • have 7-sides of an octahedron, but does not close to an 8th side

• imposing angles of 30° results in an angle difficiency about type-I vertices (240°)

Q: How can we recover constant curvature?

idea: transport curvature along edges b/w vertices.

Observation: $lk[x_3]$ has a natural map to $T_{TR}(x_3)$ given by

get arrows

• $[f_2, x_3] \longrightarrow [f_2]$

treated as a polynomial in x_1, x_2 with coefficients in $\mathbb{TR}(x_3)$

(Using different field!)

• $[f_1, f_2, x_3] \longrightarrow [f_1, f_2]$

in a similar way

$K = 12 -$ degree

$- (\#$ incoming$)$

$+ (\#$ outgoing$)$