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Harmonic quasiisometries.
w/ Yves. Benoist.

X, Y : Riemannian mfld.

$X \xrightarrow{f} Y$ smooth is HARMONIC

$\Leftrightarrow$ f is a critical point for
$E(h) = \frac{1}{2} \int |Dh|^2$

$\Leftrightarrow$ f is a tension $\tau(h) = Tr(D^2 h) = 0$

Examples:

- geodesics (constant speed)
- Riemannian immersion
  $N \xhookrightarrow{i} M$ with
  $i(N) \subset M$ minimal

Rk: harmonic $\not\Rightarrow$ isometric

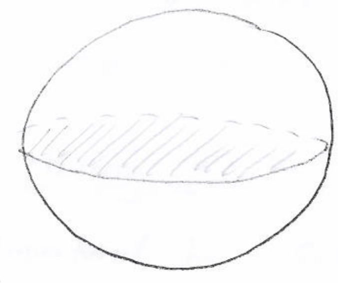
- $\mathbb{R}^n \setminus 0 \longrightarrow S^{n-1}$
  is harmonic.

a quasiisometry $X \xrightarrow{f} Y$ is a map b/w metric spaces s.t. $\exists\ c \geq 1$ s.t.

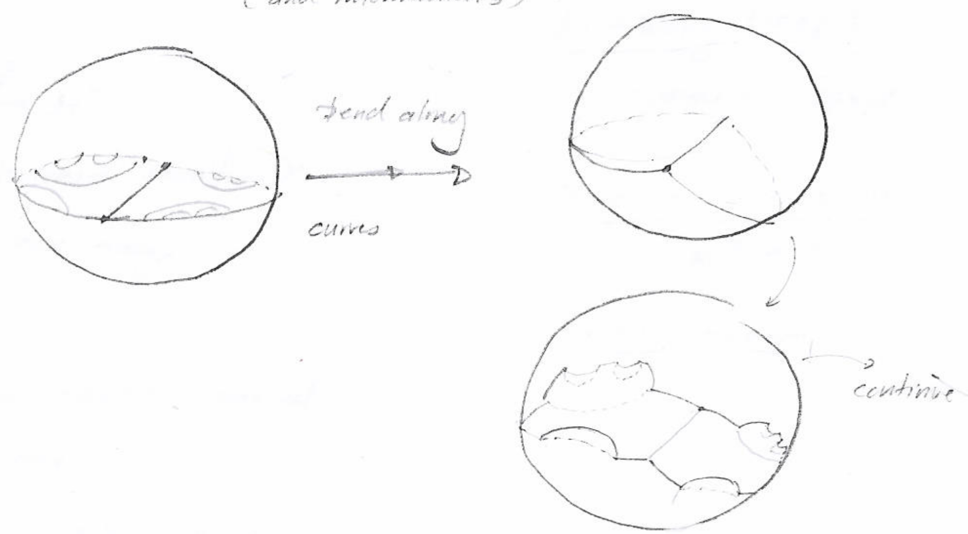
$$\frac{1}{c} d(x,y) - c \leq d(f(x), f(y)) \leq c\, d(x,y) + c.$$

Example.

- $H^2 \subset H^3$
  totally geodesic



- quasi-Fuchsian surface.
  (and intermediates)



see image by Jeff Brock.
"bug"

proof sketch:

(1) can smooth f s.t.

$|Df| \le c$, and $|D^2 f| \le c$

(2) for fixed point $x_0 \in X$, and $R > 0$

$\exists$ unique map $B_R(x_0) \xrightarrow{h_R} Y$

that is harmonic. s.t.

$h_R|_{\partial B_R(x_0)} = f|_{\partial B_R(x_0)}$

Schoen + Hamilton and Schoen + Uhlenbeck $\Longrightarrow$ $h_R \in C^\infty(\overline{B_R})$

(3) $d(h_R, f) \le M$ uniform bound.

(4) compactness result of Luckhaus $\Longrightarrow$ $h_R \longrightarrow h$ and $d(h, f) \le M$.

careful step for

$X = \mathbb{H}^2$, $Y = \mathbb{H}^3$

$\mathbb{H}^2 \xrightarrow{f} \mathbb{H}^3$

(1) for $x \in X$. smooth f to

$\tilde{f} : x \longmapsto$ "center of mass" of $f(B(x,1))$

$\alpha \in C_c^\infty((-1,1))$, $\alpha \geq 0$.

$$\Phi_x : y \in \mathbb{H}^3 \longmapsto \int_{\mathbb{H}^2} \underbrace{\alpha(d(x,z))}_{\text{mass} = 1} \cdot d(y, f(z))^2 \, dz .$$

$\tilde{f}(x) =$ the unique point $y \in \mathbb{H}^3$ s.t. Φ_x is minimal at y.

exists since.

- Φ_x is proper $\mathbb{H}^3 \longrightarrow \mathbb{R}$.
- nonpositive curvature:

  $m \in \mathbb{H}^3$ $d_m(\cdot) = d(\cdot, m)$

  d_m^2 is strictly convex , $D^2 d_m^2 \geq 2g$

here need bounded second derivatives.

Claim: $\tilde{f}$ is smooth

- use implicit function theorem.

$(x, y) \in \mathbb{H}^2 \times \mathbb{H}^3$

$$\begin{array}{c} \psi \big\downarrow \\ D_y \Phi_x \end{array} \quad \Rightarrow \quad \psi(x,y) = 0$$

when $y = f(x)$. //

remains to see $d(f, \tilde{f}) < \infty$.

$\mathbb{H}^3$ is non-positively curved.

$\Downarrow$

geodesic balls are convex

so $f(B(x,1)) \subset B(f(x), 2c)$.

end proof of ① //

proof of ③

$f : \mathbb{H}^2 \longrightarrow \mathbb{H}^3$ smooth q.i. map.

s.t. $|Df| \leq c$, $|D^2 f| \leq c$.

h_R = map on $B(x, R)$

need $d(f, h_R) \leq M$.

$\Uparrow$

Boundary estimates:

$\exists$ constant k w.r.t. c. so that

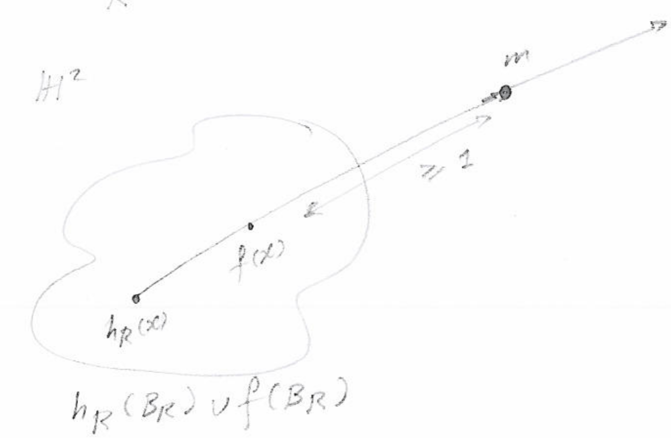
$\forall R > 0$ and $x \in B(0, R)$

$d(f(x), h_R(x)) \leq k \cdot d(x, \partial B_R)$.

Rk: need only bounds on 1st and 2nd derivatives, and not that f is a q.i. map.

pf// due to Jost.

pick geodesic ray through $h_R(x)$ and $f(x)$.



$h_R(B_R) \cup f(B_R)$

$d(f(x), h_R(x)) = d_m(h_R(x)) - d_m(f(x))$

since colinear.

Let $u : y \in \mathbb{H}^3 \longmapsto [d_m(h_R(y)) - d_m(h_R(y))]$

$- [k \cdot d(y, \partial B_R)]$

suffices to show $u(x) \leq 0$.

can show this on whole ball.

- $u = 0$ on ∂B_R by construction
- k large enough

$\Rightarrow \Delta u \leq 0$.

so

$$D^2(d_m \circ h_R) = \overbrace{D^2 d_m(Dh_R, Dh_R)}^{\geq 0} + \underbrace{D\, d_m(D^2 h_R)}_{\text{trace} \geq 0}.$$

so

$\Delta d_0 \geq 1$ in weak sense.

Thus,

$\Delta u = (\Delta \geq 0) - (|\Delta| \leq k) - (\Delta \geq k)$

$\boxtimes$