MSRI – CAT(0)

Thomas Haettel

Artin groups and nonpositive curvature

Outline:

(1) Braid groups and CAT(0) spaces
w/ D. Kulik + P. Schwer.

(2) Artin groups and CAT(0) spaces

§ 1 Braids

$n \geq 2$ (equivalent definitions)

$B_n = MCG(\text{Disc} - [n], \partial D)$

$= \pi_1\left(\left(\mathbb{C}^n - \bigcup_{i \neq j} \{z_i = z_j\}\right) / \mathrm{Sym}_n\right)$

permute coords.

= Artin presentation

$\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
$\sigma_i \sigma_j = \sigma_j \sigma_i$ otherwise $\rangle$

Q: Does B_n act geometrically on a CAT(0) space?

Kapovich + Leeb (1996)
Bridson (2009)

if $g \geq 3$

$MCG(S_g)$ does NOT act properly by semisimple isometries on a CAT(0) space.

T. Brady (2000):

from Garside structure on B_n

$\exists$ simplicial complex X_n with 1-skeleton $X_n^{(1)} = Cay(B_n)^{(1)}$ that is contractible.

Brady + McCammond (2011)

$\exists$ piecewise Euclidean metric on X_n which is CAT(0) for $n < 6$

Haettel + Kulik + Schwer (2013)

$n = 6$ is ok for above

(may also work for $n = 7$, but hard combinatorics)

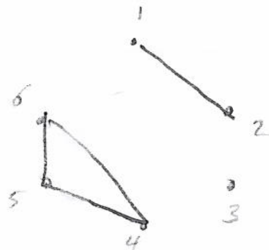
$\dim(X_n) = \operatorname{asdim}(B_n)$

<u>idea</u> : X_n is CAT(0) $\Leftrightarrow$ lk(v) is CAT(1).

L_n = suspension of the geometric realization of the poset of <u>non-crossing partitions</u> of [n]

convex hull's do not intersect in circle.

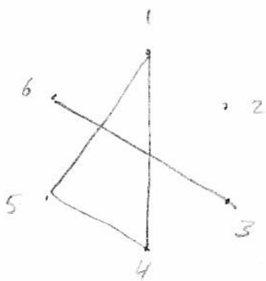
eg. <u>n=6</u>



[1 2] [3] [4 5 6].

not crossing

[1 4 5] [2] [3 6]

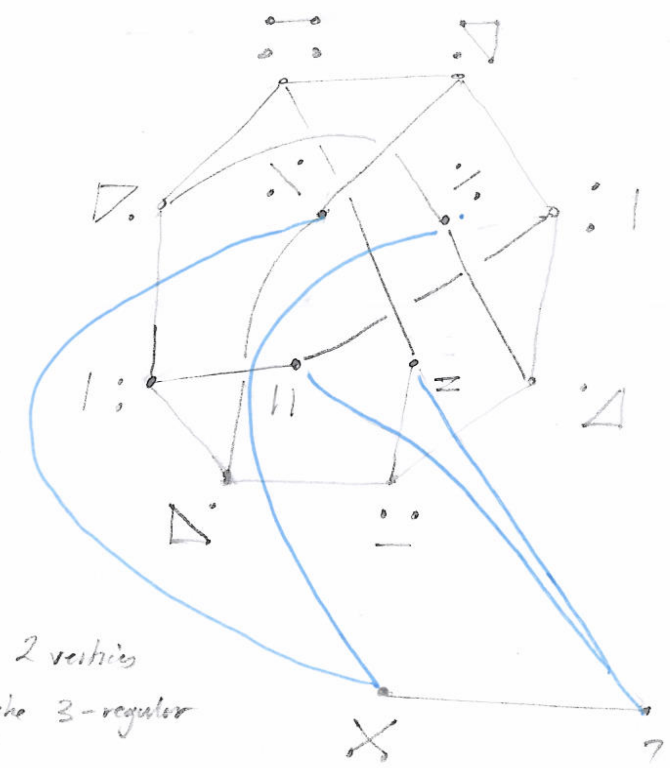


crossing

proof

<u>n=4</u>

L_4 = suspension of the graph



add 2 vertices to make 3-regular

this becomes the incidence graph of the Fano plane

is a spherical building of $PGL(3, \mathbb{F}_2)$

so in particular CAT(1)

specifically give edges length $\frac{\pi}{3}$

girth $= \frac{6\pi}{3} = 2\pi$.

$\Rightarrow$ B_4 is CAT(0).

---

§2 Artin groups

Artin groups

$$A(\Gamma) = \Big\langle v \in V(\Gamma) \;\Big|\; \underbrace{vwv\ldots}_{m_{v,w}} = \underbrace{wvw\ldots}_{m_{v,w}}, \quad v \overset{m_{v,w}}{\text{———}} w \in E(\Gamma) \Big\rangle$$

e.g. $A(\overset{3}{\triangle}{}^{3}_{2}) = B_4$

$A(\overset{2}{\text{—}}\ \cdot) = \mathbb{Z}^2 * \mathbb{Z}$

$A(>^{2}_{2}) = F_2 \times \mathbb{Z}$.

G is cubical if $G \curvearrowright X$ geometrically on a CAT(0) c.c.

$\underline{Q}$: which Artin groups are cubical.

Haettel:

$A(\Gamma)$ is cubical

$\Leftrightarrow$ (1) After removing 2-edges, the connected components of Γ are vertices, edges, even stars.

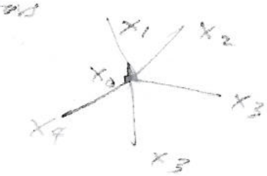
(2) $\forall v \in V(\Gamma)$, $\forall$ edges $a \overset{\text{odd}}{\text{———}} b$

$m_{va} = 2 \Leftrightarrow m_{vb} = 2$

(3) $\forall v \in V(\Gamma)$, $\forall$ stars

$\forall i \in [n]$,

$m_{vx_i} = 2 \Leftrightarrow m_{vx_j} = 2$



Corollary:

- B_n (and $B_n/Z(B_n)$) is cubical $\Leftrightarrow$ $n \le 3$. kill center
- $MCG(S_{g,p})$ is cubical $\Leftrightarrow$ $3g-3+p \le 1$
- $Out(F_n)$ is cubical $\Leftrightarrow$ $n \le 2$
- $Aut(F_n)$ is cubical $\Leftrightarrow$ $n \le 1$.

Open Q: Does $\exists$? action $B_4 \curvearrowright X$ that is proper?
X CAT(0) c.c.

Open Q: Is B_4 Haagerup?

---

proof of $B_4/Z(B_4)$ NOT cubical:

Crisp + Paoluzzi (2005)

B_3 acts properly by semisimple isometries on a CAT(0) space X

$$\Rightarrow \measuredangle_{\partial X}\left((ab)^3(+),\ b(+)\right) < \pi/2$$

visual angle

Garside elt.

attracting fixed point of $(ab)^3$.

similar.

pf// $Z(B_3) = \langle (ab)^3 \rangle \cong \mathbb{Z}$.

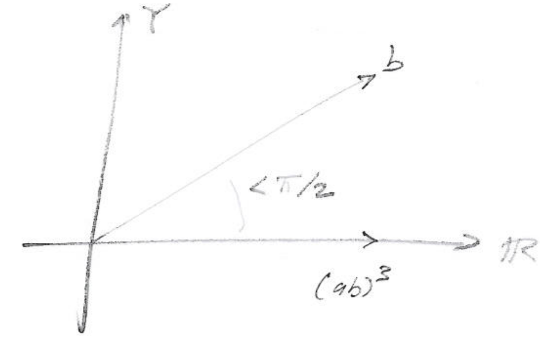
$X \cong Y \times \mathbb{R}$ where $(ab)^3$ acts by translation on $\mathbb{R}$ and trivially on Y.

If $g \in B_3$ then call $\ell(g) :=$ translation part of g on $\mathbb{R}$.

$$\ell((ab)^3) = 3\ell(a) + 3\ell(b).$$

$$a(aba) = (aba)b \Rightarrow \ell(a) = \ell(b)$$

$$\Rightarrow \ell((ab)^3) = 6\ell(b)$$



Assume $G = B_4/Z(B_4)$ acts geom. on a CAT(0) c.c. X.

Let $\{a, b, c\}$ be standard gen.s. for B_4, and $\bar{a}, \bar{b}, \bar{c}$ be their images in G.

virtually maximal abelian

$$A = \langle \bar{a}, (\bar{a}\bar{b})^3 \rangle \cong \mathbb{Z}^2 < G$$

$$B = \langle \bar{b}, (\bar{b}\bar{c})^3 \rangle \cong \mathbb{Z}^2 < G$$

Wise + Woodhouse (2015) (Cubical Flat Torus)

if $G \curvearrowright X$ CAT(0) c.c. and

$A < G$ is a virtual max'l abelian

then A is convex cocompact in X

Proposition :

$G \curvearrowright X$ CAT(0) c.c. and $A, B < G$

both convex cocompact s.t. $A \cap B$ is finite

then $\forall\ \alpha \in A$, $\beta \in B$. of infinite order.

visual angle $\measuredangle_{\partial X} (\alpha(+), \beta(+)) \geq \pi/2$.

Hence $\alpha = (\bar{a}\bar{b})^3 \in A < G$

$\beta = \bar{b} \in B < G$

$\measuredangle_{\partial X} (\alpha(+), \beta(+)) \geq \pi/2$

but $\langle \bar{a}, \bar{b} \rangle \cong B_3$ in G

so $\measuredangle_{\partial X} (\alpha(+), \beta(+)) < \frac{\pi}{2}$

Proposition If G acts geometrically

on a d-dimensional CAT(0) c.c. X

and $A < G$ abelian.

then $Z_G(A^{d!})$ is cubical.

Rk (*) this proof necessarily

fails when passing the

statement of Huang + Jankiewicz + Przytycki.

Conjecturally the virtual result

does NOT extend.