Uri Bader

Non linearity of lattices in affine buildings.

w/ Caprace + Lécureux

§ Introduction

$\Gamma_1 = \langle s, t, x \mid s^7, t^7, x^7, st = x, s^3t^3 = x^3 \rangle$

$< PGL(3, \mathbb{F}_2((t)))$

Laurent polynomials over $\mathbb{F}_2$

in fact, Γ_1 is a $PGL(3, \mathbb{F}_2((t)))$ - uniform lattice [Essert]

uniform lattice $\Rightarrow$ Properties

- $|\Gamma_1| = \infty$
- Γ_1 has (T). [Pansu, Garland]
- NST : Every proper quotients of Γ_1 are finite. Normal subgp. Thm. [Margulis]
- super rigidity.
- residually finite.

Γ_1 has a "twisted sister" Γ_2

$\Gamma_2 = \langle s, t, x \mid s^7, t^7, x^7, st = x^3, s^3t^3 = x \rangle$

Properties :

- $|\Gamma_2| = \infty$.
- Γ_2 has (T).
- Γ_2 has NST.
- every representation has finite image.

$\Rightarrow$ not linear.

conjecture : Γ_2 is virtually simple.

in fact

$$\Gamma_2 \longrightarrow \mathbb{Z}/7 \qquad x \mapsto 0, \quad s \mapsto 1, \quad t \mapsto -1$$

ker $\underset{7}{<} \Gamma$ is conjecturally simple.

Γ_1, Γ_2 are "sisters" because they are the only lattices of an $\tilde{A}_2$ - building.

§ $\tilde{A}_2$ - buildings (locally finite)

· simply connected triangle complex
s.t. $\ell(v)$ = incidence graph of
a (finite) projective plane.

Example:

analogous to Cartan's theory of symmetric spaces

Bruhat - Tits : used buildings to study
algebraic groups over a finite field.

Bruhat - Tits (BT) buildings of $PGL(3, \mathbb{F}_p((t)))$
OR
$PGL(3, \mathbb{Q}_p)$ p-adics.

since simplicial

def^n: a classical building is a
BT - building of $PGL(3, D)$
where D is a fin. dim. division
algebra over a local field.

def^n: an exotic building : are all others.
(many of them (uncountable))

Rk for $PGL(4, \cdot)$ complete correspondence
b/w symmetric spaces.

Fact: exotic buildings are a dim = 2
phenomena

eg. Morse buildings are all exotic.

Fact: $\tilde{A}_2$ buildings are CAT(0).
"Affine?"

§ Automorphisms

of classical buildings.

$$\mathrm{Aut}(\text{building}) \cong \mathrm{Aut}(PGL(3, k))$$
$$\cong PGL(3, k) \rtimes (\underbrace{\mathbb{Z}/2}_{\text{inverse transpose}} \times \underbrace{\mathrm{Aut}(k)}_{\text{automorphism of base field.}})$$

a compact group
K

Rk: $\mathrm{Aut}(\mathbb{F}_p((t)))$
is BIG
$\mathrm{Aut}(\mathbb{Q}_p)$
is finite.

of exotic buildings

- Aut might be trivial!

Conj: Aut is discrete.

Cartwright + Mantero + Steger + Zappa

have construction: cocompact + discrete Automorphism groups of exotic buildings.

Can further classify $\tilde{A}_2$-lattices as

(I) classical: lattices (virtually) contained in $PGL(3, D)$

(II) Galois: $\Gamma < PGL(3, D) \rtimes K$ with ∞-image in K.

(III) discrete groups acting cocompactly on exotic buildings

Rk type (II) may be vacuous for $PGL(3)$ examples are known for $PGL(2)$.

Γ_1 was of type (I)

Γ_2 was of type (III)

§ Properties of $\tilde{A}_2$-lattices

from link geometry

- Property (T) [Pansu, Garland, Zuk]
- NST [Margulis, Shalom + Steger]
- super-rigidity for classical (I)

unpublished, unwritten.

Hopefully can be used to show Cartwright rigidity

Main Thm: type II, III are all non-Linear.

sketch: separately work with II and III, proof is transcendental in nature

Bader + Furman: representation of ergodic actions.

Caprace + Stulemeijer: structure theory for linear loc. compact groups.

(Case II) Lie theoretic.

(case III) CAT(0) geometry

case II :

Facts ① Γ has (T) and a linear representation.

$\Rightarrow$ Γ has representation into $G(k)$,

where G: simple algebraic group over k

Zariski dense + unbounded

proof similar tools to proof of "Tits-alternative"

② Bader + Furman

$\Gamma \curvearrowright X$ ergodic and $\Gamma \xrightarrow[\text{rep}^n]{} G(k)$

then $X \longrightarrow {}^{G(k)}/_{H}$

"algebraic clothes on the naked body of X"

$\mathrm{Aut}_\Gamma(X) \longrightarrow N_{G(k)}(H)/H$

③ Caprace + Stulenmeijer :

$M^+ \triangleleft M$ linear.

M^+ simple cocompact

M loc. compact non discrete.

$\Rightarrow$ M/M^+ is v. abelian.

$\Gamma < \underbrace{PGL(3,D) \ltimes K}_{S}$, K compact

$\Gamma < K$ dense

$$T = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \mu \end{pmatrix}_{3\times 3} < S$$

$X = S/T$ is ergodic by Howe + Moore

$\Rightarrow$ $M = \mathrm{Aut}_\Gamma(X) = PGL(2,D) \ltimes K$

Γ linear $\overset{①}{\Longrightarrow}$ $\Gamma \to G(k)$

$\overset{②}{\Longrightarrow}$ $M \longrightarrow N/H$ need to show non-triviality.

$\overset{③}{\Longrightarrow}$ K is virtually abelian.

$\overset{\text{Prop. (T)}}{\Longrightarrow}$ K is finite

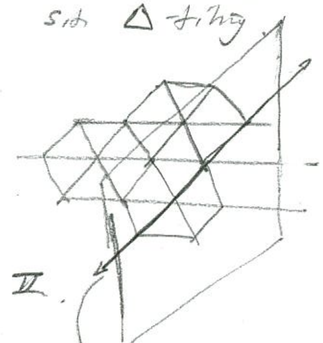
Case III :

buildings

flats tesselated by std Δ-tiling

$\rightsquigarrow$ Wall tree

tree of flats



acts like X in case II.

use geodesic flow to mimic proof of case II.

Have branching of flats.

$M =$ "group of projectivities"

acting on the wall tree

via geodesic flow

Rk: Addf + schliermach : M is NON LINEAR

for $\tilde{A}_2$ - buildings $\rightsquigarrow$ does not extend to other buildings