John Mackay

Which random groups are cubulated.

w/ Piotr Przytycki (A)

w/ Cornelia Druţu. (B)

For $m \geq 2$ (generators)

$0 < d < 1$ (density)

$l \in \mathbb{N}$ (length)

$T_{m,d,l} = \{ \langle s_1, \dots, s_m \mid R \rangle \}$

random groups

$\forall r \in R$

$|r| = l$

$|R| = (2m-1)^{l \cdot d}$

s_i, r iid

a property P holds ASYMPTOTICALLY ALMOST SURELY at density d

if $\Pr(\Gamma \in T_{m,d,l} \text{ with } P) \xrightarrow[l \to \infty]{} 1$

Gromov (1993) Ollivier (2004) (2007)

d ←——— $\frac{1}{2}$ ———→

trivial (or $\mathbb{Z}/2$) | ∞, hyperbolic non-elementary

isoperimetric inequality

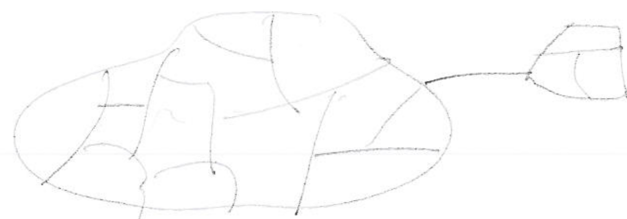
Ollivier: $\forall$ $\underbrace{\text{reduced van Kampen diagrams}}_{D}$

$$|\partial D| \geq (1 - 2d - \varepsilon) \cdot l \cdot \text{Area}(D)$$

$$\text{Cancel}(D) = \sum_{e \in D} (\deg(e) - 1)$$

[Odrzygóźdź (2014)] $\leq (d + \frac{\varepsilon}{2}) \cdot l \cdot \text{Area}(D)$

local → global can extend to generic perimeter.



asymptotically almost surely

Fact: $d < \frac{1}{12} \Rightarrow$ (a.a.s) Γ is $C'(\frac{1}{6})$

Wise (2003)

Γ f.p. $C'(\frac{1}{6}) \Rightarrow \Gamma$ acts properly and cocompactly on a CAT(0) c.c.

Ollivier + Wise (2011)

$d < \frac{1}{6} \underset{a.a.s.}{\Rightarrow} \Gamma$

$\Rightarrow \Gamma$ is a-T-menable

stronger than linear

(acts properly on a Hilbert space)

$d < \frac{1}{5} \Rightarrow \Gamma$ acts nontrivially on a CAT(0) c.c.

$\Rightarrow \Gamma$ acts on a Hilbert space with unbounded orbit

(NOT property (T))

Zuk (2003) Kotowski + Kotowski (2012):

$d > \frac{1}{3} \underset{a.a.s.}{\Rightarrow}$ Every action of Γ on a Hilbert space has a fixed point.

Rk property (T) is monotone

that is $G \twoheadrightarrow H$ and G has (T)

$\Rightarrow H$ has (T) too.

Bollobas + Thomason (1987):

$\exists\ \chi_m^*(L)$ s.t. if $\chi_m(L) = |R|$ (num. of relations)

and $\dfrac{\chi_m(L)}{\chi_m^*(L)} \xrightarrow[L\to\infty]{} \infty \Rightarrow \Gamma$ has (T)

$\dfrac{\chi_m(L)}{\chi_m^*(L)} \xrightarrow[L\to\infty]{} 0 \Rightarrow \Gamma$ does NOT have (T).

We know that

$$(2m-1)^{(\frac{1}{5}-\varepsilon)L} \le \chi_m^*(L) \le (2m-1)^{(\frac{1}{3}+\varepsilon)L}$$

$\forall\ \varepsilon > 0$

Main Thm (A) :

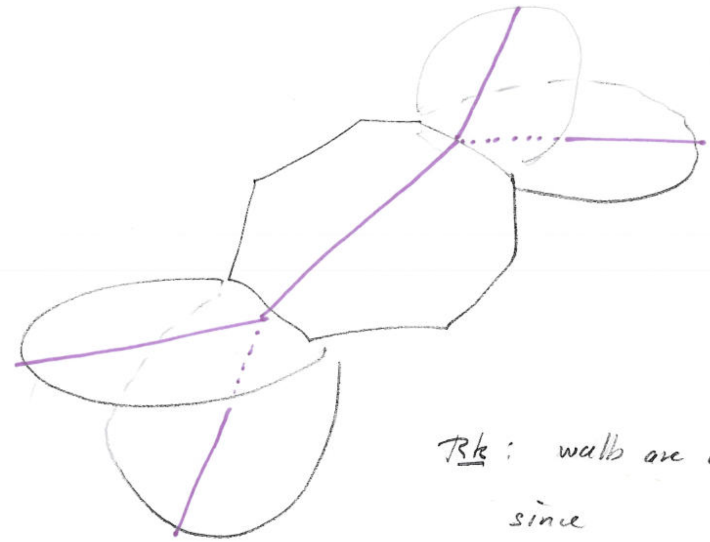
$d < \frac{5}{24} \implies \Gamma$ does not have (T).

Main Thm. (B)

$\exists\, c > 0$ s.t.

$\chi_m^*(L) \leq c \cdot L\,(2m-1)^{\frac{1}{3}L}$

$\implies \Gamma$ has (T).



Rk: walls are embedded

since

$\text{cancel}(D) < (d+\epsilon)\, L \cdot \text{Area}(D)$

can avoid this via coarse hyperbolicity

Loose this for $d > \frac{1}{5}$.

Idea of Ollivier + Wise.

$\Gamma \curvearrowright X$ cayley complex.

build a "wall"

locally want to be an embedded tree.

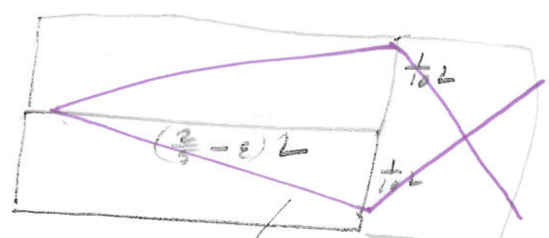
$H = \text{Stab}(\text{wall})$

$\implies \text{end}(\Gamma, H) > 1$.

quotient.

Sageev construction $\implies$ have CAT(0) c.c.

Problem when $d = \frac{1}{5}$

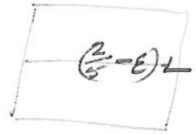


cancel $= \frac{3}{5}L$
$< d \cdot 3L$
ok

allowed + very misbehaved fills complex.

To fix this for Thm (A)

use <u>TILES</u> T



subcomplexes with large overlap.

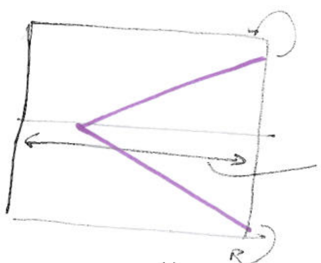
$$\text{Cancel}(T) > \frac{1}{4}(|T|-1)L$$

<u>Note</u>: if $d < \frac{1}{4}\left(\frac{N}{N+1}\right)$

then $|T| \leq N$

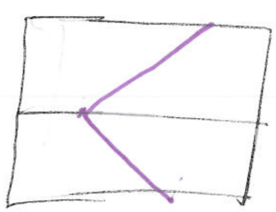
so $d < \frac{5}{24} \Rightarrow |T| \leq 5$

solution: <u>rewire walls in tile</u>



$> \frac{1}{4}$ of perimeter

$\Downarrow$



this fixes problem.

towards Thm (B)

does not mention property (T) in paper.

~~Garland~~, <u>Pansu</u>, <u>Zuk</u>, <u>Ballmann + Swiatkowski</u>,

<u>Gromov</u> (for good measure)

$$\Gamma = \langle s_1, \ldots, s_m \mid r_1, \ldots, r_n \rangle$$

and $|r_i| = 3$ triangular presentation.

$\Gamma \curvearrowright X$ ← cayley complex.

$\forall x \in X$, $\lambda_1(lk(x)) > \frac{1}{2}$

1st Eigenvalue of graph Laplacian.

$\Rightarrow \Gamma$ has property (T)

<u>Note</u> $$\lambda_1(lk(x)) = \inf\left(\frac{\sum_{e \in lk} |df(e)|^2}{\inf_{c \in \mathbb{R}} \sum_{v \in lk} |f(v)-c|^2 \deg(v)}\right)$$

d-regular

f non-constant
$lk^{(x)} \longrightarrow \mathbb{R}$

$$\Delta = I - \frac{1}{\deg} A.$$

adjacency

idea of proof of (3)

say $\Gamma \curvearrowright H$ Hilbert space.

$\mathcal{F} = \{ f : \Gamma \to H \ : \ \Gamma\text{-equivariant} \}$

$$E(f) = \sum_{i=1}^{m} |f(s_i) - f(e)|^2 \deg(s_i)$$

if $\exists f \in \mathcal{F}$ s.t. $E(f) = 0 \Rightarrow$ done

"minimal energy."

"makes sense since working with random graphs"

Ultralimits $\rightsquigarrow$ $\inf E(f)$ is attained.

$$E(f) = \sum_{i=1}^{m} | f(s_i) - f(e) |^2 \deg(s_i)$$

$$= \frac{1}{2} \sum_{x \sim e} |f(x) - f(e)|^2 \deg(\underset{x}{\bullet}\!\overset{e}{\text{---}}\!\bullet)$$

$$= \frac{1}{2} \inf_{c \sim H} \sum_{x \in L^0} |f(x) - c|^2 \deg(x)$$

$$\leq \frac{1}{2\lambda_1(L)} \sum_{e \in L'} |df(e)|^2 = \frac{1}{2\lambda_1(L)} E(f)$$

idea (Zuk)

$\langle s_1, \ldots, s_m \mid s_j^{\pm 1} \pi_1^{i}(s_j^{\pm 1}) \pi_2^{i}(s_j^{\pm 1}) \rangle$

link at identity $\longleftrightarrow$ union of random permutation graphs.

$K - K = \Gamma = \langle s_1, \ldots, s_m \mid R \rangle$

look at

$\hat{\Gamma} = \langle T \mid \hat{R} \rangle$

words in S_i of length $\frac{L}{3}$.