Anne Thomas

main result.

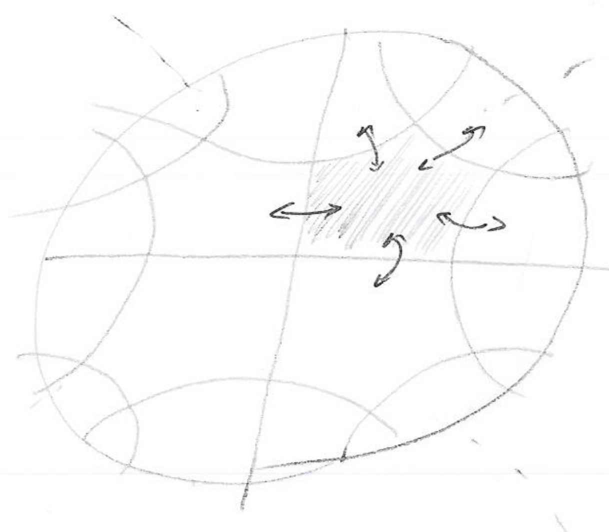
Quasi-isometry and commensurability classification of certain right-angled Coxeter groups

w/ Palavi Dani (A)

P. Dani + E. Stark (B)

RACG
Geometric amalgams of free groups

W() =



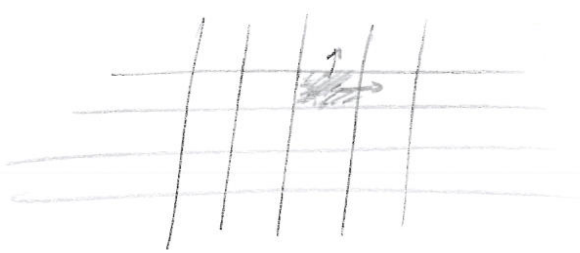
G, H commensurable
⇓ ⇑ only for some groups
quasi-isometric. 3-mfld gps [Schwartz]

Poincaré Polygon Thm

W(Polygon) = tesselation.

view W()

as reflections in sides



Davis + Januszkiewicz.

RACG
|— finite.
RAAG

converse fails!
hyperbolic ⟺ no edges ⟺ free
∃ 1-ended RACG.

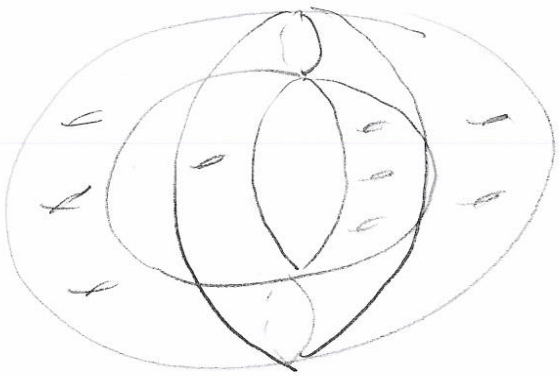
Geometric amalgams (Lafont)

surface amalgams

- ∘ surface points

or ∘ ≥ 3 surfaces (branching)

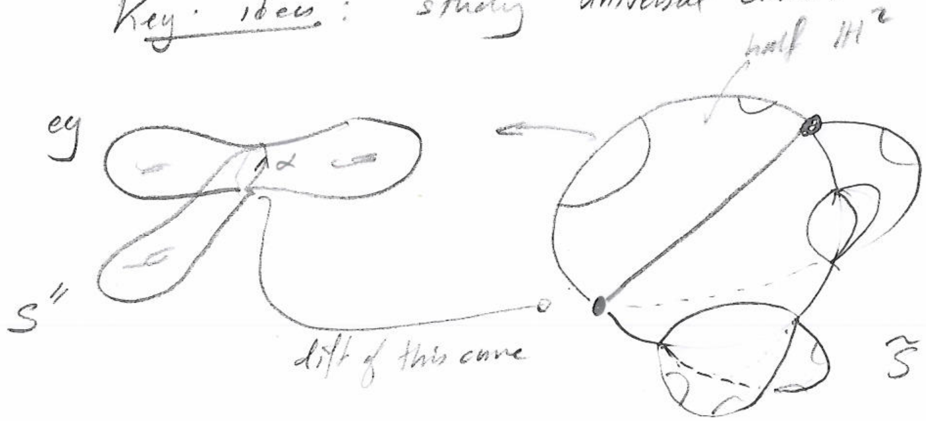
eg



look at π_1 (these surfaces)

Key idea: study universal cover

eg



Bowditch's JSJ ~~tree~~ graph

understanding boundary

by looking at pairs of points

and how they cut

the boundary

$G = \pi_1(S)$.

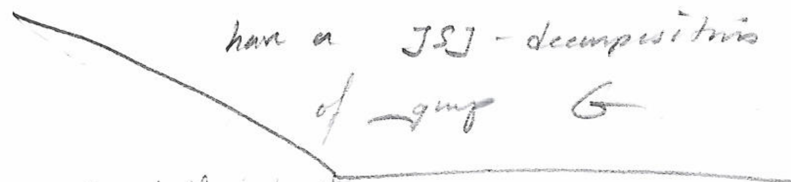
$G \curvearrowright \partial\tilde{S} \Rightarrow G \curvearrowright (\text{JSJ tree})$

Labelling cells (verts., edges)

by their stabilizers

have a JSJ-decomposition

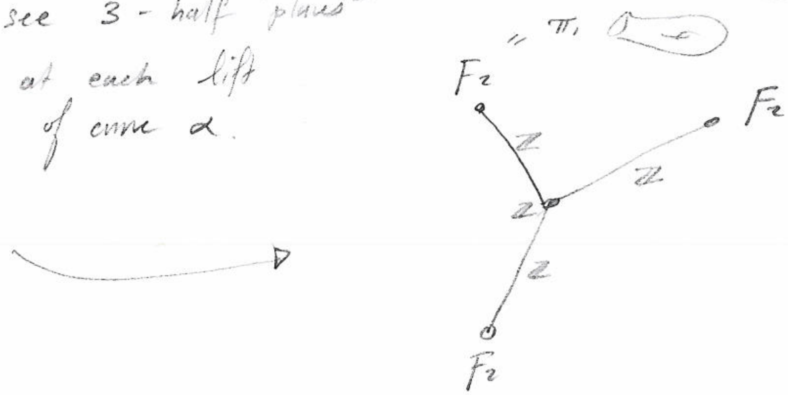
of group G



see 3-half planes

at each lift

of curve α.



To use JSJ-decomposition for Coxeter groups need to restrict to subclass of RACGs] Eg

Thm (A): (visual construction)

(1) construction of JSJ-decomp. from geometric properties of graph.

(2) No  minor.

Rk JSJ (Coxeter) is always a *tree*

Why? cycle $\Rightarrow$ HNN

$\Rightarrow G \twoheadrightarrow \mathbb{Z}$

but Coxeter is generated by 2-torsion $\Rightarrow$ cannot surject $\mathbb{Z}$!

- Δ-free, 3-convex ← only to simplify statements
- 1-ended, hyp., no cocompact Fuchsian.
- no  minor.

Rk: all these examples are covered by surface amalgams.

Crisp + Paoluzzi: can understand W_{min} (augmented RACG's) as $W_{min} = \pi_1$ (orbicomplex).

edges $\longleftrightarrow$ mirrors.

* Lafont: $\pi_1\left(\underset{S_1}{\text{surface amalgam}}\right) = \pi_1(S_2)$

$\Rightarrow S_1 \cong S_2$ via $\chi(S_1) = \chi(S_2)$

to extend Lafont's result
want to have G of Davis complex
but fails to have homeo. b/w graphs.

to build geometric amalgam
using tiling in same style
as work of Futer + Thomas
on hyperbolic buildings.

to see commensurability
can load vectors with
· Euler characteristic of pieces
and compare vectors.

once have geometric amalgams
covering orbicomplexes
find common homeomorphic
cover using Lafont.

Rk: JSJ - decomp
is q.i. - invariant

Compare to result of
J. Huang : characterization
of q.i. invariance of RAAGs.
commensurable $\Leftrightarrow$ q.i. $\Leftrightarrow$ $\Gamma \cong \Gamma'$
very different here.

Quasi-isometry and commensurability classification of certain right-angled Coxeter groups

Anne Thomas

School of Mathematics and Statistics, University of Sydney

Groups acting on CAT(0) spaces
MSRI
30 September 2016

Outline

1. Quasi-isometry and commensurability
2. Groups
   - Right-angled Coxeter groups (RACGs)
   - Geometric amalgams of free groups
3. Quasi-isometry classification of certain RACGs
   with Pallavi Dani
4. Commensurability classification of certain RACGs and geometric amalgams of free groups
   with Pallavi Dani and Emily Stark

Quasi-isometry and commensurability

G, H finitely generated groups

Write $G \sim_{QI} H$ if quasi-isometric

G and H are (abstractly) commensurable, denoted $G \sim_{AC} H$,
if $\exists$ finite index subgroups $G' < G$ and $H' < H$ with $G' \cong H'$

$G \sim_{AC} H \implies G \sim_{QI} H$

Converse holds for some classes of groups e.g. fundamental groups of hyperbolic 3-manifolds with boundary (Fraggerio)

RAAGs and RACGs

Γ finite simplicial graph with vertex set S

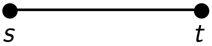
The right-angled Artin group (RAAG) associated to Γ is

$$A_\Gamma = \langle S \mid st = ts \iff s \text{ and } t \text{ are adjacent in } \Gamma \rangle$$

The right-angled Coxeter group (RACG) associated to Γ is

$$W_\Gamma = \langle S \mid st = ts \iff s \text{ and } t \text{ are adjacent in } \Gamma, \text{ and } s^2 = 1 \forall s \in S \rangle$$

Examples of RACGs

$\Gamma_1 =$ 

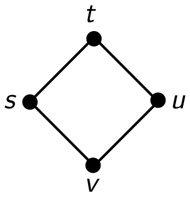
$\Gamma_2 =$ 

- $W_{\Gamma_1} = \langle s, t \mid s^2 = t^2 = 1 \text{ and } st = ts \rangle \cong C_2 \times C_2$ finite

- $W_{\Gamma_2} = \langle s, t \mid s^2 = t^2 = 1 \rangle \cong D_\infty$ which is 2-ended

Examples of RACGs

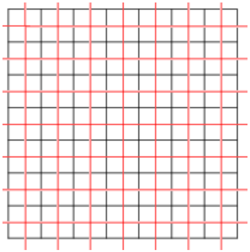
Γ a 4-cycle



then

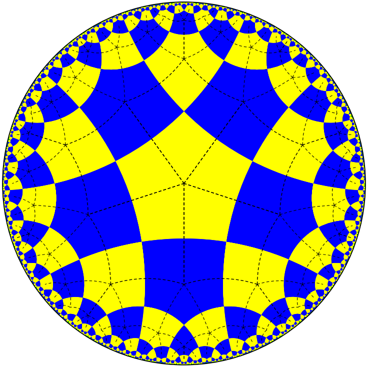
$$W_\Gamma = \langle s, t, u, v \rangle = \langle s, u \rangle \times \langle t, v \rangle \cong D_\infty \times D_\infty$$

is group generated by reflections in sides of square



Examples of RACGs

If $\Gamma = 5$-cycle, W_Γ is group generated by reflections in sides of right-angled hyperbolic pentagon:

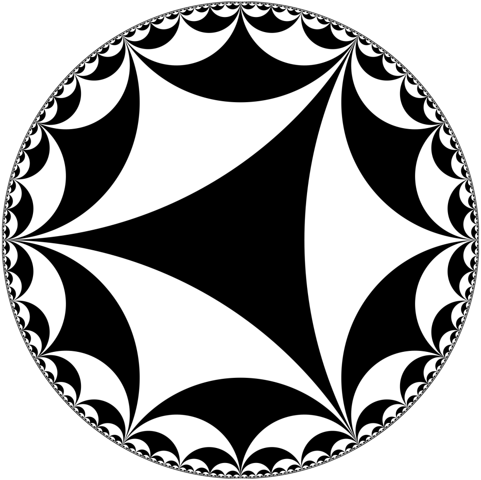


Similarly if Γ is an n-cycle, $n \geq 5$.

(Thanks to Jon McCammond for the picture.)

Examples of RACGs

W_Γ can act on hyperbolic plane properly but not cocompactly
e.g. if Γ is 3 vertices, no edges



W_Γ is virtually free

Relationship between RAAGs and RACGs

Theorem (Davis–Januszkiewicz)

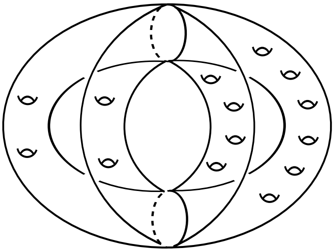
Every RAAG is commensurable to a RACG.

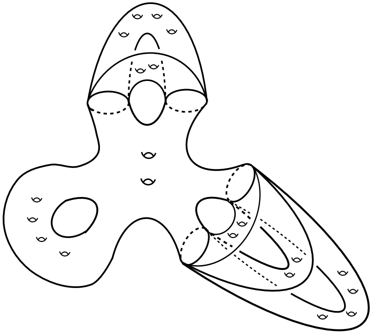
The converse is not true:

- A_Γ hyperbolic $\iff$ Γ has no edges $\iff$ A_Γ free but $\exists$ 1-ended hyperbolic RACGs
- By considering divergence, $\exists$ infinitely many QI classes of W_Γ which are not QI classes of any RAAG (Behrstock–Charney, Abrams–Brady–Dani–Duchin–Young, Dani–T)

Geometric amalgams of free groups

Some surface amalgams:





A geometric amalgams of free groups is the fundamental group of a surface amalgam. Introduced by Lafont.

Consider universal covers.

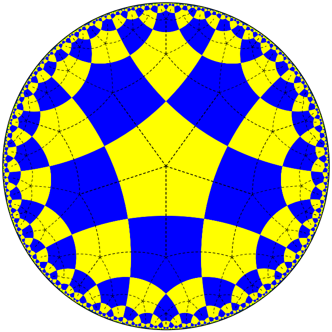
Bowditch's JSJ tree

Bowditch's JSJ tree is a QI invariant for 1-ended hyperbolic groups which are NOT cocompact Fuchsian.

G is cocompact Fuchsian if it acts geometrically on $\mathbb{H}^2$.

Fact

W_Γ is cocompact Fuchsian $\iff$ $W_\Gamma = W_{\Gamma'} \times W_{\Gamma''}$ where Γ' is n-cycle, $n \geq 5$, and $W_{\Gamma''}$ is finite.



Bowditch's JSJ tree

Bowditch's JSJ tree T_G uses topological features of ∂G:

- $x \in \partial G$ is a local cut point of valence $k \geq 2$ if $\exists$ nbhd U of x s.t. $U \setminus \{x\}$ has k components.
- $\{x, y\} \subset \partial G$ is a cut pair of valence $k \geq 2$ if x, y are local cut points of valence k, and $\partial G \setminus \{x, y\}$ has k components.

Vertices of T_G:

- Type 1 (finite valence). Cut pairs in ∂G of valence ≥ 3. These account for all local cut points of valence ≥ 3 in ∂G.
- Type 2 (quadratically hanging). Equiv. classes in ∂G of local cut points of valence 2, with $x \sim y$ if $x = y$ or $\{x, y\}$ is cut pair of valence 2.
  Equiv. classes $\simeq$ Cantor sets in $\partial \mathbb{H}^2$, with Type 1 vertices in their closure.
- Type 3. These "fill in the gaps" to form a tree.

$G \sim_{QI} H \implies \exists$ type-preserving isomorphism $T_G \to T_H$

Bowditch's JSJ tree

$G \curvearrowright \partial G$ induces $G \curvearrowright T_G$

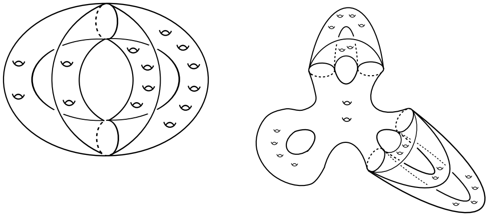
Quotient is finite graph called the JSJ graph of G

Edge stabilisers are maximal 2-ended subgroups over which G splits

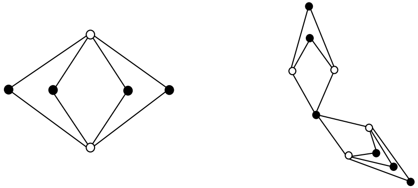
Induced graph of groups is the JSJ decomposition of G

Examples of JSJ decompositions

Geometric amalgams of free groups:



JSJ graphs: Type 1 vertices white, Type 2 vertices black



JSJ decompositions: $\mathbb{Z}$ on white vertices and all edges, free gps on black vertices.

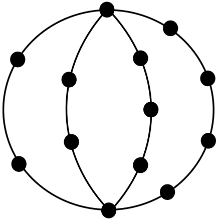
Visual construction of Bowditch's JSJ tree for certain W_Γ

We restrict to Γ triangle-free, and assume

- W_Γ is 1-ended $\iff$ Γ is connected and has no separating vertices or edges (Davis)
- W_Γ is hyperbolic $\iff$ Γ has no 4-cycles (Moussong)
- W_Γ not cocompact Fuchsian $\iff$ Γ is not an n-cycle, $n \geq 5$
- W_Γ splits over a 2-ended subgroup $\iff$ Γ has a cut pair of vertices (Mihalik–Tschantz)

Examples

Generalised Θ-graphs



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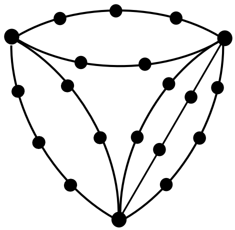
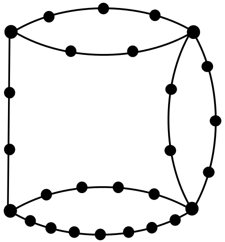
Examples

Cycles of generalised Θ-graphs





Visual construction of Bowditch's JSJ tree for certain W_Γ

Theorem (Dani–T 2016)

Assume Γ is triangle-free and W_Γ is 1-ended, hyperbolic, not cocompact Fuchsian, and splits over a 2-ended subgroup.

1. *We give a "visual" construction of the JSJ decomposition of W_Γ i.e. construct the JSJ graph and its vertex and edge groups in terms of subsets of S given by explicit graph-theoretic criteria.*
2. *We prove that for the subclass where Γ has no K_4 minor, Bowditch's JSJ tree $T_\Gamma = T_{W_\Gamma}$ has no Type 3 vertices.*

Using part 2 and work of Gromov, Bowditch, Cashen:

Corollary

For Γ, Γ' as above with no K_4 minors, the following are equivalent:

1. $W_\Gamma \sim_{QI} W_{\Gamma'}$
2. $\partial W_\Gamma \simeq \partial W_{\Gamma'}$
3. $\exists$ *type-preserving isomorphism* $T_\Gamma \to T_{\Gamma'}$

Special case: JSJ decomposition for $\Gamma \in \mathcal{G}$

A vertex of Γ is essential if it has valence ≥ 3, and Γ is 3-convex if each path between essential vertices has at least 3 edges.

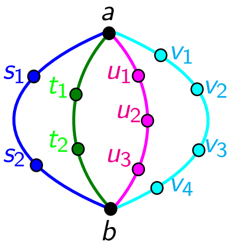
Write $\mathcal{G}$ for the class of graphs Γ so that:

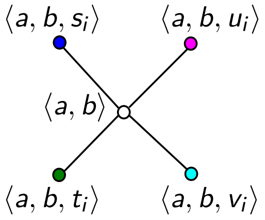
- Γ is triangle-free and 3-convex (simplifying assumptions)
- W_Γ is 1-ended, hyperbolic, not cocompact Fuchsian, and admits a splitting over 2-ended subgroup (Bowditch's JSJ tree is a QI invariant and is nontrivial)
- Γ has no K_4 minor (Bowditch's JSJ tree is a complete QI invariant).

Special case: JSJ decomposition for $\Gamma \in \mathcal{G}$

If $\Gamma \in \mathcal{G}$ then the JSJ decomposition of W_Γ has:

- a Type 1 vertex of valence $k \geq 3$ for each essential cut pair $\{a, b\} \subset S$ of valence $k \geq 3$. The vertex group is $\langle a, b \rangle$.
- a Type 2 vertex for each maximal $A \subset S$ s.t. elements of A pairwise separate $|\Gamma|$, with $\langle A \rangle$ infinite and not 2-ended. This vertex has valence = # pairs in A giving Type 1 vertices. The vertex group is $\langle A \rangle$.
- Edge between Type 1 vertex and Type 2 vertex $\iff$ their stabilisers intersect.

$\Gamma =$ 



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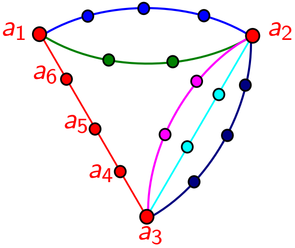
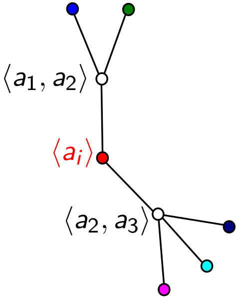




Commensurability results

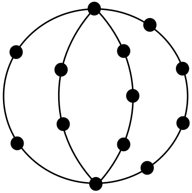
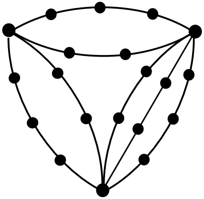
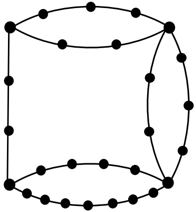
Assume from now on that $\Gamma \in \mathcal{G}$ i.e. Γ is triangle-free and 3-convex, and Bowditch's JSJ tree is a complete QI invariant.

Theorem (Dani–Stark–T 2016)

We give explicit necessary and sufficient conditions for commensurability of all W_Γ where $\Gamma \in \mathcal{G}$ is either:

- *a generalised Θ-graph, or*
- *a cycle of generalised Θ-graphs.*

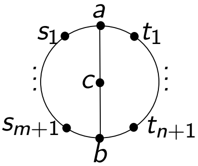
That is, we give the commensurability classification of all W_Γ with $\Gamma \in \mathcal{G}$ so that the JSJ graph of W_Γ is a tree of diameter ≤ 4.

Previous results on commensurability for Coxeter groups

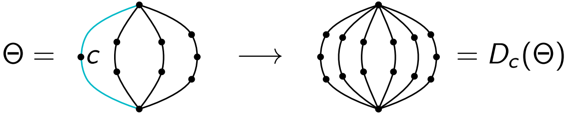
Theorem (Crisp–Paoluzzi 2008)

For $m \geq n \geq 1$ let $W_{m,n}$ be defined by:



Then $W_{m,n} \sim_{AC} W_{k,\ell} \iff \frac{m}{n} = \frac{k}{\ell}$.

We recover this result, and give commensurability invariants for all generalised Θ-graphs, not just the 3-convex ones, by doubling:

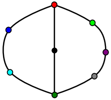
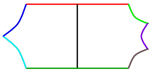


The map $c \mapsto 1$, $s \mapsto 0$ for $s \in S \setminus \{c\}$ gives

$$1 \to W_{D_c(\Theta)} \to W_\Theta \to \mathbb{Z}/2\mathbb{Z} \to 1$$

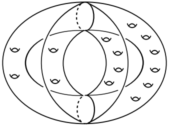
Crisp–Paoluzzi proof strategy

$W_{m,n}$ is fundamental group of hyperbolic orbicomplex $\mathcal{O}_{m,n}$

$W_{m,n} \sim_{AC} W_{k,l} \implies \exists$ torsion-free, finite-index $G < W_{m,n}$, $H < W_{k,l}$ with $G \cong H$.

G and H are geometric amalgams of free groups e.g.



Theorem (Lafont 2007)

If $\mathcal{X}$ and $\mathcal{X}'$ are surface amalgams, any isomorphism $\pi_1(\mathcal{X}) \to \pi_1(\mathcal{X}')$ is induced by a homeomorphism $f : \mathcal{X} \to \mathcal{X}'$.

Analyse this homeo to get Euler characteristic necessary conditions.

Topological rigidity

Key result used by Crisp–Paoluzzi: topological rigidity for surface amalgams i.e.

Theorem (Lafont 2007)

If $\mathcal{X}$ and $\mathcal{X}'$ are surface amalgams, any isomorphism $\pi_1(\mathcal{X}) \to \pi_1(\mathcal{X}')$ is induced by a homeomorphism $f : \mathcal{X} \to \mathcal{X}'$.

For general W_Γ with $\Gamma \in \mathcal{G}$, the usual geometric realisation is the Davis complex Σ_Γ i.e. Cayley graph with squares filled in.

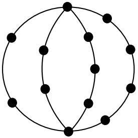
Theorem (Stark 2016)

Topological rigidity does NOT hold for quotients of Σ_Γ: there is a cycle of generalised Θ-graphs Γ and torsion-free, finite-index subgroups $G, G' < W_\Gamma$ s.t. $\pi_1(\Sigma_\Gamma/G) \cong \pi_1(\Sigma_\Gamma/G')$ but Σ_Γ/G and Σ_Γ/G' are NOT homeomorphic.

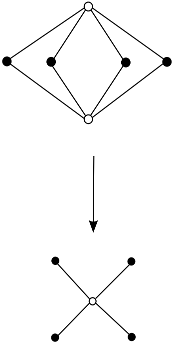
RACGs and geometric amalgams of free groups

We construct a new geometric realisation of W_Γ for all $\Gamma \in \mathcal{G}$: a piecewise hyperbolic orbicomplex $\mathcal{O}_\Gamma$ with $\pi_1(\mathcal{O}_\Gamma) = W_\Gamma$, s.t. any torsion-free, finite-sheeted cover of $\mathcal{O}_\Gamma$ is a surface amalgam.

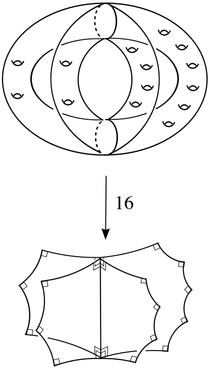
Construction of $\mathcal{O}_\Gamma$ uses JSJ decomposition from Dani–T.



Γ



JSJ graphs



$\mathcal{O}_\Gamma$

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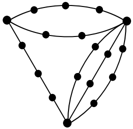
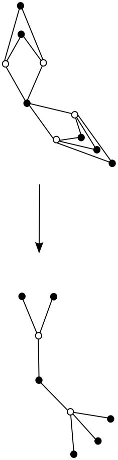
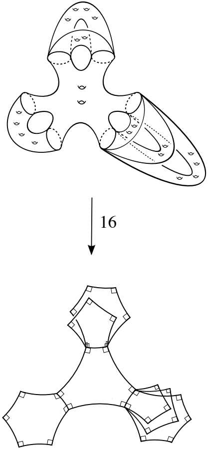


Γ



JSJ graphs



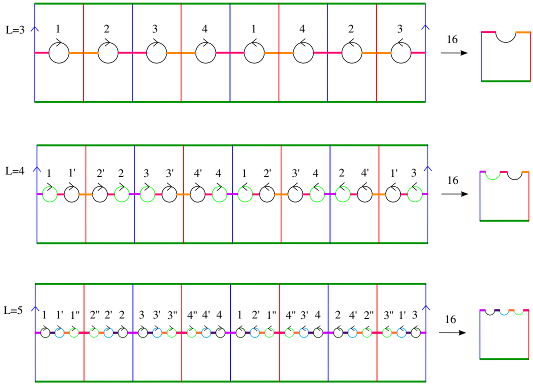
$\mathcal{O}_\Gamma$

RACGs and geometric amalgams of free groups

Theorem (DST)

For all $\Gamma \in \mathcal{G}$, the group W_Γ has an index 16 subgroup which is a geometric amalgam of free groups.

We construct a surface amalgam which 16-fold covers $\mathcal{O}_\Gamma$ by tiling surfaces with 2 boundary components "nicely" with 16 right-angled hyperbolic polygons. Tilings are similar to those in Futer–T.



RACGs and geometric amalgams of free groups

Theorem (DST)

For all $\Gamma \in \mathcal{G}$, the group W_Γ has an index 16 subgroup which is a geometric amalgam of free groups.

Using similar ideas, and well-known results on coverings of surfaces with boundary, we also prove:

Theorem (DST)

If a geometric amalgam of free groups has JSJ graph a tree, it is commensurable to some W_Γ (with $\Gamma \in \mathcal{G}$).

Corollary

Commensurability classification of geometric amalgams of free groups with JSJ graph a tree of diameter ≤ 4.

This generalises a result of Stark, who considered surface amalgams obtained by gluing S, S' along essential curve in each.

Commensurability invariants

Each induced subgraph Λ of Γ has corresponding special subgroup $W_\Lambda = W_A = \langle A \rangle$ where A is vertex set of Λ.

Our commensurability invariants are families of equations involving the Euler characteristics of special subgroups:

$$\chi(W_\Lambda) := 1 - \frac{\#V(\Lambda)}{2} + \frac{\#E(\Lambda)}{4}$$

Result for generalised Θ-graphs

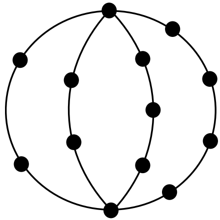
Theorem (DST)

Let Γ, Γ' be 3-convex generalised Θ-graphs with branches $\beta_1, \dots, \beta_n$ and $\beta'_1, \dots, \beta'_{n'}$ respectively, ordered so that

$$\chi(W_{\beta_1}) \geq \cdots \geq \chi(W_{\beta_n}) \quad \text{and} \quad \chi(W_{\beta'_1}) \geq \cdots \geq \chi(W_{\beta'_{n'}}).$$

Then $W_\Gamma \sim_{AC} W_{\Gamma'} \iff n = n'$ and $\exists K, K' \in \mathbb{Z}$ s.t. $\forall i$

$$K' \cdot \chi(W_{\beta_i}) = K \cdot \chi(W_{\beta'_i})$$



Result for generalised Θ-graphs

Theorem (DST)

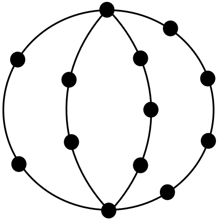
Let Γ, Γ' be 3-convex generalised Θ-graphs with branches $\beta_1, \ldots, \beta_n$ and $\beta'_1, \ldots, \beta'_{n'}$ respectively, ordered so that

$$\chi(W_{\beta_1}) \geq \cdots \geq \chi(W_{\beta_n}) \quad \text{and} \quad \chi(W_{\beta'_1}) \geq \cdots \geq \chi(W_{\beta'_{n'}}).$$

Then $W_\Gamma \sim_{AC} W_{\Gamma'} \iff$ the vectors

$$v = (\chi(W_{\beta_1}), \ldots, \chi(W_{\beta_n})) \quad \text{and} \quad v' = (\chi(W_{\beta'_1}), \ldots, \chi(W_{\beta'_{n'}}))$$

are commensurable.



Theorem (DST)

Let Γ, Γ' be 3-convex cycles of N, N' generalised Θ-graphs, s.t. the central vertices in their JSJ decomp. have stabilisers $W_A, W_{A'}$. Then $W_\Gamma \sim_{AC} W_{\Gamma'}$ iff at least one of the following holds:

1. *The Euler char. vectors of the gen. Θ-graphs in Γ and in Γ' have the same commens. classes, and for each such commens. class, containing $\{v_i \mid i \in I\}$ and $\{v'_i \mid i' \in I'\}$,*

$$\chi(W_{A'}) \sum_{i \in I} \overline{v_i} = \chi(W_A) \sum_{i' \in I'} \overline{v_{i'}}$$

   where $\overline{v_i} = \sum_j \chi(W_{\beta_{ij}}) =$ sum of entries of v_i, similarly for $v_{i'}$.

2. *All nontrivial gen. Θ-graphs in Γ, Γ' have $r \geq 2$ branches, all Euler char. vectors for Γ (resp. Γ') are commens., and w and w' are commens., where w formed by putting entries of*

$$(\sum_i \chi(W_{\beta_{i1}}), \sum_i \chi(W_{\beta_{i2}}), \ldots, \sum_i \chi(W_{\beta_{ir}}), \chi(W_A))$$

   in non-decreasing order, similarly for w'.

Necessary and sufficient conditions for commensurability

Suppose Γ, Γ' are both 3-convex generalised Θ-graphs, or both 3-convex cycles of generalised Θ-graphs.

Necessary conditions: $W_\Gamma \sim_{AC} W_{\Gamma'} \implies \exists$ surface amalgams $\mathcal{X} \to \mathcal{O}_\Gamma$ and $\mathcal{X}' \to \mathcal{O}_{\Gamma'}$ with $\pi_1(\mathcal{X}) \cong \pi_1(\mathcal{X}')$.
Analyse Lafont homeomorphism $f : \mathcal{X} \to \mathcal{X}'$ to deduce information on Euler characteristics of surfaces, hence of special subgroups.

Sufficient conditions: If our Euler characteristic conditions hold, pass to our degree 16 covers, then construct common finite covers of these.

For generalised Θ-graphs, necessary conditions proof is adaptation of Stark, sufficient conditions proof is similar to Crisp–Paoluzzi.

For cycles of generalised Θ-graphs, both directions are delicate, and make heavy use of Γ, Γ' having similar structure.

Quasi-isometry vs commensurability

Using our commensurability classification and work of Dani–T and Malone on JSJ trees:

Corollary

Every QI class of a group we consider contains infinitely many abstract commensurability classes.

Compare to Huang's results for certain RAAGs:
$A_\Gamma \sim_{QI} A_{\Gamma'} \iff A_\Gamma \sim_{AC} A_{\Gamma'} \iff A_\Gamma \cong A_{\Gamma'}$.

Questions

- Which geometric amalgams of free groups are commensurable to right-angled Coxeter groups, and vice versa?
- Is there a finite list of "moves" on defining graphs so that $W_\Gamma \sim_{AC} W_{\Gamma'} \iff \Gamma$ and Γ' are related by a finite sequence of such moves? Holds for generalised Θ-graphs (Crisp–Paoluzzi arguments), certain RAAGs (Huang).
- Is there a class of defining graphs so that for Γ, Γ' in this class, $W_\Gamma \sim_{QI} W_{\Gamma'} \iff W_\Gamma \sim_{AC} W_{\Gamma'}$?