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SYZ mirror symmetry in the complement of a divisor & regular functions on the mirror

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§1. SYZ mirror symmetry
§2. Examples
§3. Divisors, potentials

Mirror symmetry:

symplectic geometry of X $\longleftrightarrow$ algebraic geometry on mirror $X^\vee$

First for Calabi-Yau, global holomorphic vol. form.

Homological mirror symmetry (Kontsevich '94)

Fukaya cat $\mathcal{F}(X)$ $\xleftrightarrow[\text{equiv.}]{\text{derived}}$ $Coh(X^\vee)$

Lagrangian submflds (+ local systems) $\longleftrightarrow$ coherent sheaves

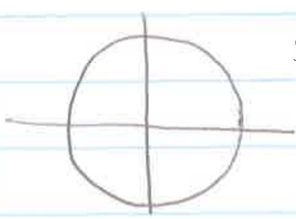
Lagrangian Flow homology holom. discs with boundary on Lag's $\longleftrightarrow$ morphisms of sheaves sheaf cohomology ring sheaves

Stroninger - Yau - Zaslow conj (1996)
Mirror pairs of Calabi-Yau carry dual Lagrangian form fibrations.

Symplectic manifold : (X, ω), closed non-deg, 2 form
locally $\simeq \left(\mathbb{R}^{2n}, \sum_{\substack{x_1, \dots x_n \\ y_1, \dots y_n}} dx_i \wedge dy_i\right)$

Lagrangian: a manifold $L^n \subset X^{2n}$, $\omega|_L \equiv 0$
locally think of as
$\underset{x_1, \dots x_n}{\mathbb{R}^n} \subset \underset{\substack{x_1, \dots x_n \\ y_1, \dots y_n}}{\mathbb{R}^{2n}}$

Ex : $\mathbb{C} = \mathbb{R}^2$

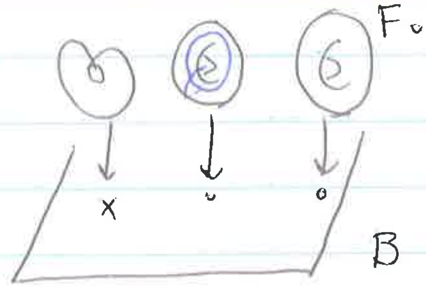


$S^1(r)$

$\mathbb{C}^n \supset S^1(r_1) \times \cdots \times S^1(r_n)$

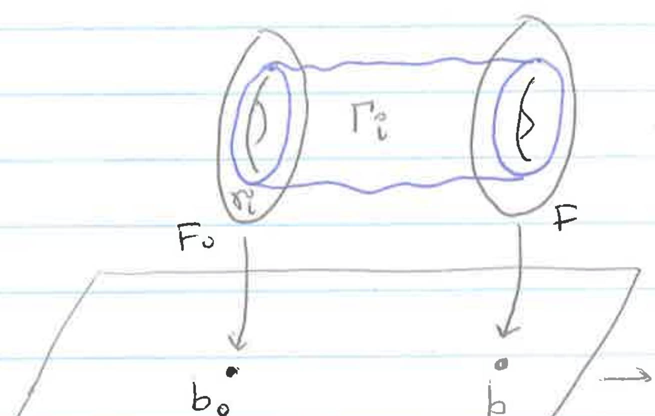
$X^{2n} \subset Y$

$$\begin{array}{ccccc} F_0 & \underset{\text{Lagr. Torus}}{\subset} & X^{2n} & \subset & Y \\ \downarrow & & \downarrow \pi & & \\ b_0 & \in & B & & \\ & & \uparrow & & \\ & & \text{affine structure} & & \end{array}$$



Neighborhood of $b_0 \in B$ is $\cong$ nbd of 0 in $H^1(F_0, \mathbb{R})$ $(\cong \mathbb{R}^n)$

Pick a basis $\gamma_1, \dots, \gamma_n$ of $H_1(F_0)$



$\rightsquigarrow \int_{\Gamma_i} \omega \in \mathbb{R}$

(measures how much γ_i has moved)

$\rightarrow \left(\int_{\Gamma_1} \omega, \dots, \int_{\Gamma_n} \omega \right) \in \mathbb{R}^n$

☆ move $b_0 \longrightarrow$ translations in $\mathbb{R}^n$

☆ chg basis of $H_1(F_0) \rightsquigarrow GL(n, \mathbb{Z})$

B has an integral affine structure these charts glue via $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$

Dual Torus : $F^\vee = H^1(F, U(1))$
$= \hom(\pi_1 F, U(1))$
$= \{U(1) \text{ local systems on } F\}$

OR if $F = V/\Gamma$ then $F^\vee = V^*/\Gamma^\vee$

Outside of sing-fibers, dual fibration :

$$\begin{array}{ccc} F^\vee & \longrightarrow & X^\vee = \{(b \in B, \nabla \in \hom(\pi_1 F_b, U(1))\} \\ & & \downarrow \\ & & B \quad (\text{smooth}) \end{array}$$

Complex coordinates on $X^\vee$ locally :

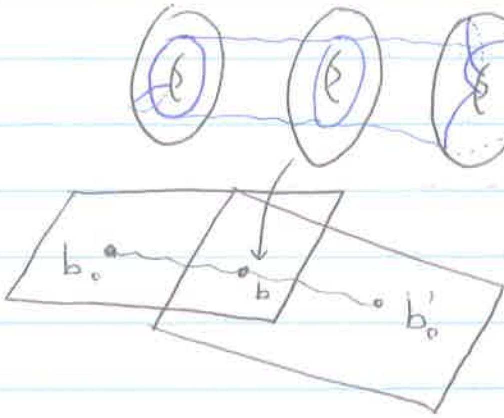
$$z_i(b, \nabla) = \underbrace{\exp\left(-\int_{\Gamma_i} \omega\right)}_{\mathbb{R}_+} \cdot \underbrace{\nabla(\gamma_i)}_{U(1)} \in \mathbb{C}^*$$

$X^\vee$ complex manifold

Slightly better way :
to the extent convergence is OK
build $X^\vee$ by gluing together charts $((\mathbb{C}^*)^n, (z_1, \dots, z_n))$

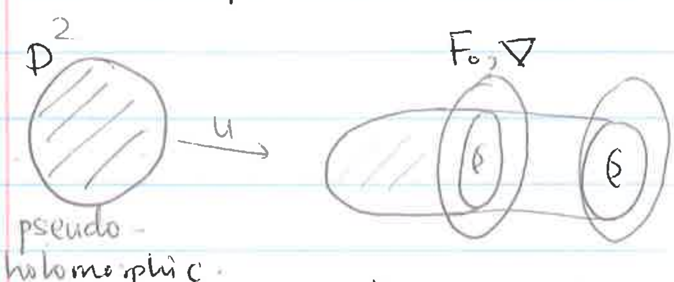
$$\begin{array}{ccccc} (b, \nabla) & \in & X^\vee & & (z_1, \dots, z_n) \\ \downarrow & & \downarrow \pi^\vee & & \downarrow \\ b & & B & & (-\log|z_1|, \dots, -\log|z_n|) \end{array}$$

Gluing $\longrightarrow$



Glue by
- ~~reset~~ rescaling
- monomial transformations

Fact: Quantities that come up in Lagrangian Floer theory of (F, ∇) are holomorphic in the (z_i) [holomorphic as functions on $\{(F,\nabla)\} = X^\vee$]

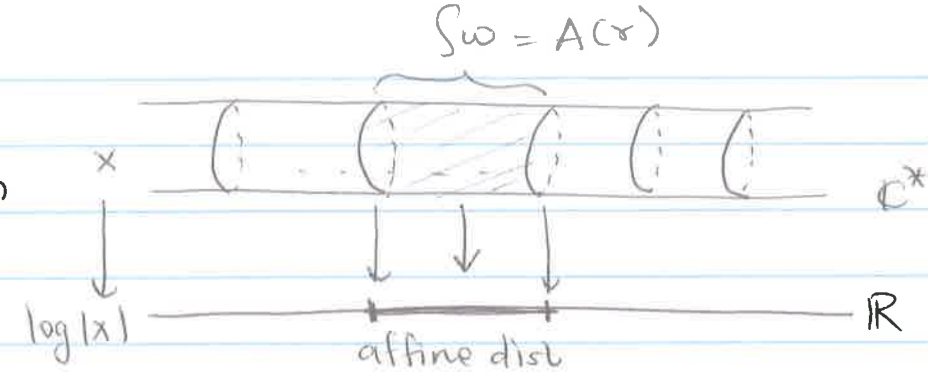


disc counts with weight
$\exp(-\text{area}) \cdot \nabla(\partial u)$
$=$ const. monomial in (z_i)

§2. Examples:

1) $X = \mathbb{C}^*$

ω = area form



mirror $X^\vee = \mathbb{C}^*$ coord $z = e^{-\mathrm{Area}(\gamma)} \nabla(\gamma)$

1') $X = (\mathbb{C}^*)^n$, product tori $\rightsquigarrow X^\vee = (\mathbb{C}^*)^n$.

2) $X = \mathbb{C}^2 \setminus \{xy = 1\}$ std. sympl. of $\mathbb{C}^2$

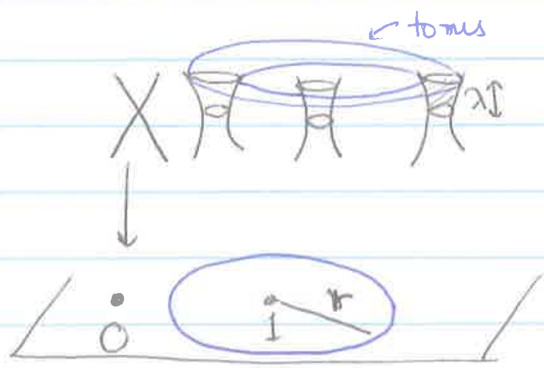
$$(x,y) \longmapsto (\underbrace{\log|xy-1|}_{=\log r}, \underbrace{\pi(|x|^2 - |y|^2)}_{=\lambda})$$

$X \longrightarrow B = \mathbb{R}^2$

$(x,y) \in \mathbb{C}^2$

$\downarrow$

$xy \in \mathbb{C}$



$(xe^{i\theta}, ye^{-i\theta})$

$T_{r,\lambda}$: Lagr. Tori

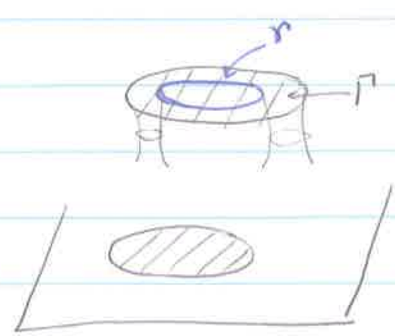
$T_{1,0}$ = 

pinched torus.

Local coords on $X^\vee$?

$[\gamma_0]: w(T, \nabla) = e^{-\lambda} \nabla(\gamma_0)$

$[\gamma]: z(T, \nabla) = e^{-\int_\gamma \omega} \nabla(\gamma)$



Fine for $r < 1$

For $r > 1$, 2 natural cond's for "γ"

$r \gg 1$

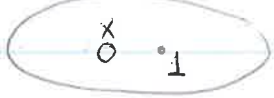
Product torus in $\mathbb{C}^2$

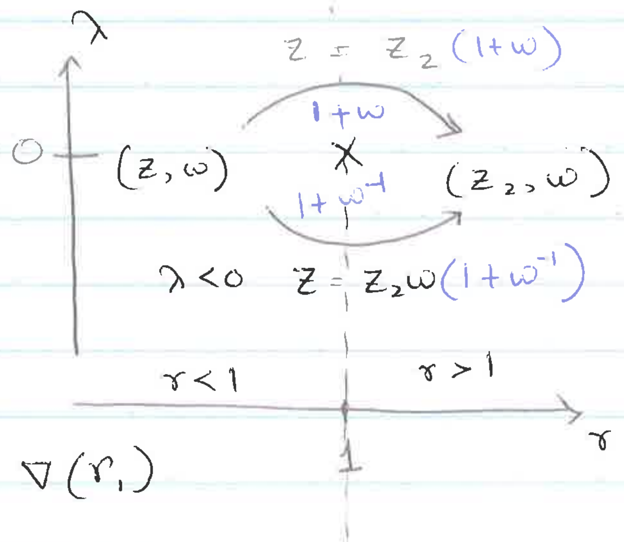
$\cong$

$S^1(a) \times S^1(b) \subset \mathbb{C}^2$

$D^2(a) \times pt$

$pt \times D^2(b)$





$$z_1 = e^{-\int_{\Gamma_1} \omega} \nabla(\Gamma_1)$$

$$z_2 = e^{-\int_{\Gamma_2} \omega} \nabla(\Gamma_2)$$

$$z_1 = z_2 w$$

As cross "wall" $r = 1$,

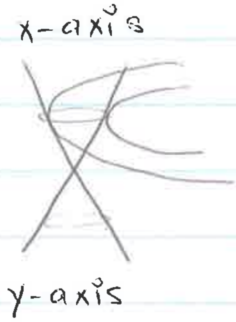
$T_{1,\lambda}$ bounds holom disc $\subset$ coord axis

If $\lambda > 0$: area $= \lambda$

$\partial = \Gamma_0$

$\lambda < 0$: area $= -\lambda$

$\partial = -\Gamma_0$



Mirror is actually $(\mathbb{C}^*)^2_{z,w} \cup (\mathbb{C}^*)^2_{z_2,w} \cup pt$

$z = z_2(1+w)$

$\mathbb{C}^2 \setminus \{uz = 1\}$ $\quad u = z_2^{-1}$

$uz = z_2^{-1} z = 1 + w$

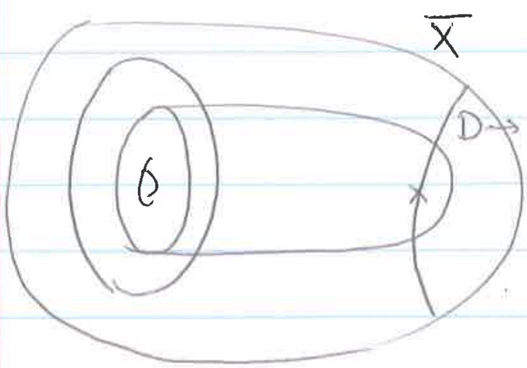
(Partial) Compactification:

$$X \underset{\text{open}}{\subset} Y \qquad\qquad X^\vee$$

$$\begin{array}{l} \cap \\ \overline{X} \rightsquigarrow W_{\overline{X}} \in \mathcal{O}(X^\vee) \end{array}$$

add to X normal crossings divisor $|D| \in c_1(\overline{X})$ (chern class)

$X = \overline{X} \setminus D$



$(F, \nabla) =$ pt in $X^\vee$

→ holo. disc

Take $\sum_{\text{such discs}} e^{-\text{Area}} \nabla(\partial u) = W_{(F,D)}$

$X = \mathbb{C}^2 \setminus \{xy = 1\} \subset \mathbb{C}^2$

$$W_{\mathbb{C}^2} = \begin{cases} z & r < 1 \\ z_2 + z_2 w & r > 1 \end{cases}$$