Cluster algebras and cluster varieties 3/28/16

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Outline:

- cluster algebra + examples + properties
- Cluster $\mathcal{A}$ - variety
- Cluster $\mathcal{X}$ - variety

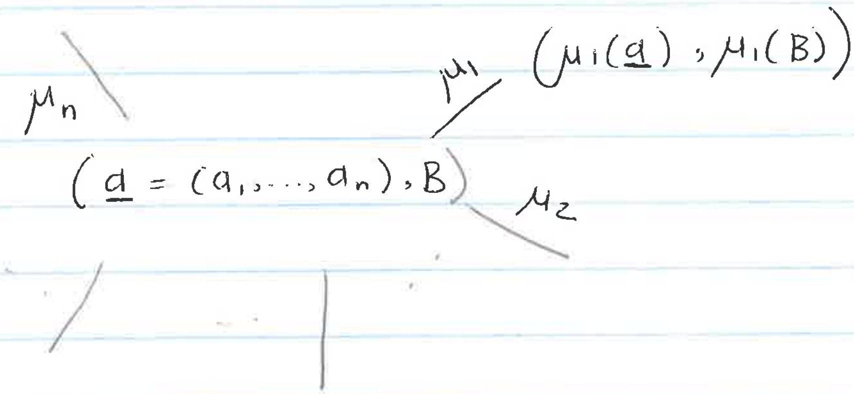
"Def": A cluste algebra $\mathcal{A}$ is a subring of $k(a_1, \ldots, a_n)$, field of rational functions in $\{a_1, \ldots, a_n\}$.

Generators constructed by series of exchange relations

Def^n: A Seed Σ for $\mathcal{A}$ is an initial cluster $\underline{a} = (a_1, \ldots, a_n)$ plus $n \times n$ skew-symmetrizable integral matrix B.

($\exists$ $d_i > 0$ s.t. $d_i b_{ij} = -d_j b_{ji}$ $\forall$ i, j)

From each seed, can mutate in each of n directions, obtaining n more seeds:



Mutation:

Given seed $\Sigma = (\underline{a} = (a_1, \dots, a_n), B = (b_{ij}))$, define new seed $\mu_k(\Sigma) = (\mu_k(\underline{a}), \mu_k(B))$:

Cluster $\mu_k(\underline{a}) := (a_1, \dots, a_{k-1}, a_k', a_{k+1}, \dots, a_n)$ where a_k' defined by (1) $a_k a_k' = \prod_{b_{ik}>0} a_i^{b_{ik}} + \prod_{b_{ik}<0} a_i^{|b_{ik}|}$

$\mu_k(B)$ defined by:

$$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} & \text{if } b_{ik} b_{kj} \leq 0 \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \end{cases}$$

Note: $\mu_k^2 = \mathrm{id}$ & $\mu_k(B)$ is again skew symmetrizable.

To define cluster algebra:

Start from initial seed Σ & apply all possible sequences of mutations.

This produces set of all cluster variables

Def^n: Cluster alg $\mathcal{A} = \mathcal{A}(\underline{a}, B)$ is subring of $k(a_1, \dots, a_n)$ generated by all cluster variables

Note: sometimes designate in advance that we only will mutate in directions $1, \dots, l$, $l < n$. Then call $a_{l+1}, \dots, a_n$ frozen variables

Ex:

Rank 2 clust. alg.

$\mathcal{F} = \mathbb{Q}(a_1, a_2)$. Fix $b, c \in \mathbb{N}$ & let $B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$

Initial seed: $\Sigma = ((a_1, a_2), B)$

Apply μ_1: new variable a_1' defined by

$$a_1 a_1' = a_2^c + 1 \leftarrow \text{empty product}$$

& new matrix: $\mu_1(B) = -B$

Apply mutation μ_2: new variable a_2':

$$a_2 a_2' = 1 + a_1^b$$

$\uparrow$ empty product

& new matrix: $\mu_2(B) = -B$

Refer to a_1' as a_3 & a_2' as a_0

$$\xrightarrow{\mu_1} ((a_1, a_0), -B) \xrightarrow{\mu_2} ((a_1, a_2), B) \xrightarrow{\mu_1} ((a_3, a_2), -B) \xrightarrow{\mu_2} \cdots$$

In general have clust var's $\{a_m\}_{m \in \mathbb{Z}}$ defined by
$$a_{m-1} a_{m+1} = \begin{cases} a_m^b + 1 & m \text{ odd} \\ a_m^c + 1 & m \text{ even} \end{cases}$$

$\mathcal{A} = \mathcal{A}((a_1, a_2), B)$ is a subring of $\mathcal{F}$ gen. by a_m's.

By construction, each a_m can be written as rational exp in a_1 & a_2.

Th^m (F&Z)* Any clust. var can be written as Laurent polynomial in variables of a given cluster.

* F&Z: Fomin & Zelevinsky

Ex. Let $b = c = 1$. Start with (a_1, a_2)

$$a_3 = \frac{a_2 + 1}{a_1}$$

$$a_4 = \frac{a_3 + 1}{a_2} = \frac{\frac{a_2+1}{a_1} + 1}{a_2} = \frac{1 + a_1 + a_2}{a_1 a_2}$$

$$a_5 = \frac{a_4 + 1}{a_3} = \frac{\frac{1+a_1+a_2}{a_1 a_2} + 1}{\left(\frac{a_2+1}{a_1}\right)} = \frac{a_1 + 1}{a_2}$$

$$a_6 = a_1 \,!$$

Positivity conj. (now thm)

Let $\underline{a}$ be any cluster in clust. alg $\mathcal{A}$ & let a' be any cluster variable. Then a' can be expressed as Laurent poly in var's of a with pos. coeff's.

B is skew symm: Lee Schiffler

General case: GHKK.

Def^n: A cluster alg has finite type if there are only finitely many clust var's.

Th^m: (F & Z) Such cluster alg's classified by Dynkin diagrams.

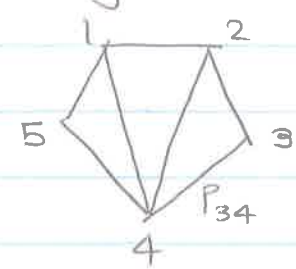
✯ Cluster $\mathcal{A}$-varieties.

We have seeds & mutations as before: Associate torus $\mathcal{A}_\Sigma = (\mathbb{C}^*)^n$ to each seed $\Sigma = (\underline{a}, B)$. Interpret clust var's $\underline{a}$ as the standard coords on torus $\mathcal{A}_\Sigma$.
If $\mu_k(\Sigma) = \Sigma'$, then the map μ_k on clust var's $\rightsquigarrow$ birational iso $\mu_k : \mathcal{A}_\Sigma \to \mathcal{A}_{\Sigma'}$
The <u>cluster $\mathcal{A}$-variety</u> $\mathcal{A}_{[\Sigma]}$ is scheme over $\mathbb{Z}$ obtained by gluing all seed $\mathcal{A}$-tori for seeds equivalent to taking affine closure.

Ex. Grassmannian $Gr_{2n}(\mathbb{C}) = \{ V \subset \mathbb{C}^n \mid \dim V = 2 \}$
Elements represented by full rank $2 \times n$ matrices M. For $1 \le i \le j \le n$,

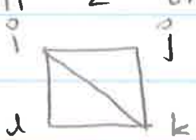
$$p_{ij}(M) = \det \begin{pmatrix} m_{1i} & m_{1j} \\ m_{2i} & m_{2j} \end{pmatrix}$$

Given triangulation T of n-gon, we get seed $\Sigma(\ )$
Cluster is

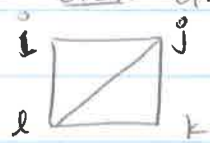


$$(p_{14}, p_{24}, \underbrace{p_{12}, p_{23}, p_{34}, p_{45}, p_{15}}_{\text{frozen}})$$

Cluster torus is where these clust var's are $\neq 0$
If 2 triangulations ~~defi~~ differ by <u>flip</u>, then



flip



the corre. clust tori glued using exchange relation $p_{ik}\, p_{jl} = p_{ij}\, p_{kl} + p_{jk}\, p_{il}$

Cluster X-variety: We have seeds & mutations as before.

Obtained by gluing together tori, but now our coord's on tori denoted $X_1, \ldots, X_n$ &

③ $$\mu_k(X_i) = \begin{cases} X_k^{-1} & \text{if } i=k \\ X_i\left(1+X_k^{-\operatorname{sgn}(b_{ik})}\right)^{-b_{ik}} & \text{if } i \neq k \end{cases}$$

where $$\operatorname{sgn}(\ell) = \begin{cases} -1 & \text{if } \ell < 0 \\ 0 & \text{if } \ell = 0 \\ 1 & \text{if } \ell > 0 \end{cases}$$

Note: Fock-Goncharov: sometimes think of seeds as $(\Lambda, \{e_i\}, (*,*))$

lattices w/ dist bases ; bilinear form 'skew symmetrizable'

This form encodes matrix $B = (b_{ij})$ i.e.

$$b_{ij} = (e_i, e_j)$$

Then we define mutation on basis $\{e_i\}$ by:

$\mu_k(\{e_i\}) = \{\tilde{e}_i\}$ where

$$\{\tilde{e}_i\} = \begin{cases} e_i + \max((e_i, e_k), 0)\, e_k & \text{if } i \neq k \\ -e_k & \text{if } i = k \end{cases}$$

Exercise:

If $B' = \mu_k(B)$ as in (2), then $b'_{ij} = (\tilde{e}_i, \tilde{e}_j)$

$\square$