Theta functions for Log Calabi-Yau manifolds I

- Sean Keel.

Ex. Y = 2 dim affine space $\cong \mathbb{C}^2_{x,y}$

$\mathcal{O}(Y)$: function on $Y \cong \mathbb{C}[x,y]$

$B_Y = \{x^a y^b \mid a,b \geq 0\}$

Nothing canonical

$\{(x+y^b)^a y^b\}$ just as good.

Let's take 2 lines

$U \cong (\mathbb{C}^*)^2$

$\mathcal{O}(U) \cong \mathbb{C}[x,y,x^{-1},y^{-1}]$

$B_U = \{x^a y^b \mid a,b \in \mathbb{Z}\} \cong \mathbb{Z}^2$

Lemma : B_U is canonical!

Lemma: $f \in \mathcal{O}(U)$

$f = \lambda x^a y^b$, $\lambda \in \mathbb{C}^*$

$$\begin{array}{ccc} \mathcal{O}(U) & \supset & \mathcal{O}(Y) \\ \cup & & \\ B_U & \supset & B_U \cap \mathcal{O}(Y) = B_Y \end{array}$$

Once we choose $U \subset Y$, get a basis B_Y which is canonical.

(This can be vastly generalized)

What's special about U?

U is log Calabi-Yau (affine with max bdry)

$\Rightarrow$ unique ~~vot~~ vol form

$U \subset Y$ is partial min. "P.M.M"

new ω has pole along $D := \partial U = Y \setminus U$

U smooth (Y, D) : normal crossing $U \subset \overline{Y}$

$$H^0(Y, \omega_Y(D)^{\otimes M}) \subset H^0(U, \omega_U^{\otimes M})$$

$\uparrow$ log forms $\qquad\qquad$ $\uparrow$ depends only on U.

Def^n U is Log Calabi Yau if these vec spaces are all 1-dim. Basis $\omega^{\otimes m}$, ω : volume form.

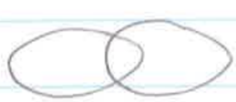
Non Ex: $\mathbb{A}^1_z$, dz has double pole at ∞

$\therefore$ non calabi-Yau

$$\underset{\substack{\| \\ (\mathbb{C}^*)^N}}{(\mathbb{G}_m)^N_{z_1, \dots, z_n}} \quad \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n} = \omega \quad \therefore \log CY$$

Prop: $T^2 \overset{\text{open}}{\subset} Y \overset{\text{P.M.M}}{\longleftrightarrow} Y$ is T-toric

Let $T_1 \underset{\text{Birat}}{\overset{\varphi}{\dashleftarrow\dashrightarrow}} T_2$ log CY



$$\overset{T_1}{\cap}\overset{T_2}{\cap} \hspace{-2em} U = T_1 \cup_\varphi T_2$$

U is log CY $\longleftrightarrow$ $\varphi^*(\omega) = \omega$

All cluster varieties are examples

Let $Y = G$ $\quad$ e.g. $G = SL_n(\mathbb{C})$

$G \circlearrowright Y \therefore G \circlearrowright \mathcal{O}(Y)$

$\therefore \mathcal{O}(Y) = \bigoplus_\lambda V_\lambda \otimes V_\lambda^*$

would love : a canonical basis

$\Rightarrow$ canonical basis of all rep V_λ (impossible)

G is too symm, no can. fun^n at all.

If choose $H \subset \overset{\text{Borel}}{B} \subset G$

$\Rightarrow$ $U \subset Y = G$ $\leftarrow$ Lustig found Basis of $\mathcal{O}(G)$

$\uparrow$ open double bruhat

U is often log CY max boundary.

& U is cluster variety.

In fact this will give first ex of clusters.

Def^n: U log CY

$U^{trop}(\mathbb{Z}) = \left\{ (E, N) \;\middle|\; \begin{array}{l} E \ \partial\text{-division in some} \\ \text{min. model} \end{array} \right\} \cup \{0\}$

$$\begin{array}{l} E_1 \\ \cap \\ Y_1 \\ \updownarrow \text{ biratl } \varphi \\ Y_2 \supset E_2 \end{array} \quad U \subset Y_1,\ U \subset Y_2$$

$E_1 = E_2$ here $\leftrightarrow$ φ extends on E_1 & E_2 is an iso.

$\leftrightarrow$ $U \subset Y \supset E$

$V_E : Q(U) \setminus \{0\} \longrightarrow \mathbb{Z}$

$f \longmapsto \mathrm{ord}_E(f)$
$\quad$ " pole

$\mathrm{ord}_E(f+g) = \min(\mathrm{ord}(f), \mathrm{ord}(g))$.

$U^{top}(\mathbb{Z}) \subset U^{top}(\mathbb{R})$

e.g. $(\mathbb{C}^*)^N = T_L := L \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^N$

$L \cong \mathbb{Z}^N$ $\quad$ $L =$ co-char

Lemma $\quad T_L^{trop}(\mathbb{Z}) \subset T_L^{trop}(\mathbb{R})$

$$\begin{array}{ccc} T_L^{trop}(\mathbb{Z}) & \subset & T_L^{trop}(\mathbb{R}) \\ \| \text{ canonically} & & \cap \\ L & \subset & L_{\mathbb{R}} \end{array}$$

Suppose $T \overset{\text{open}}{\subset} U$ Both log CY

$\Rightarrow T^{trop}(\mathbb{Z}) = U^{trop}(\mathbb{Z})$ $\qquad \omega|_T = \omega_T$

e.g. If $T = T_L$, identify $U^{trop}(\mathbb{Z}) = L \cong \mathbb{Z}^N$

Warning: $U^{trop}(\mathbb{Z})$ not a group.

$v_1 + v_2$ is not a valuation

$$\begin{array}{lll} U \supset T_L & \rightsquigarrow \; U^{trop} \sim L_{\mathbb{R}} & \\ & \qquad\qquad \downarrow \text{iso} & \leftarrow \text{not linear} \\ \quad \cup\, T_L & \qquad \sim \; L_{\mathbb{R}} & \quad\;\; \text{piecewise linear.} \end{array}$$

Say (Y, D) normal crossing Min. model

$\;\;\;\;\uparrow$ compact.

$U \subset Y$, $\Sigma_{(Y,D)} \subset \mathbb{R}^D \leftarrow$ once ~~…~~ coord for each $E \subset D$

$\downarrow$

irred compo. D_i

rays $P_E = P_{D_i}$

Span $(P_{E}, \ldots, P_{E})$ in $\Sigma_{(Y,D)}$

iff $E_1 \cap E_2 \cap \cdots \cap E_D \neq \phi$ in Y

Fact: $|\Sigma_{(Y,D)}|$ depends only on U

$\;\;\;\;\;\;\;\; \overset{\shortparallel}{U^{trop}(\mathbb{R})}$

e.g. $U = T_L \subset Y \leftarrow$ toric

$\Sigma_{(Y,D)} =$ usual fan.

See: $U^{trop}(\mathbb{Z})$ gen co-char of a torus

How about the character?

"Co-characters" of "The dual".

max boundary

Def^n: max $\partial \longleftrightarrow \dim_{\mathbb{R}} U^{trop}(\mathbb{R}) = \dim_{\mathbb{C}} U$

$(\underset{\text{torus}}{T} \subset U \Rightarrow \text{max } \partial)$

Mirror construction:

Fix (Y, D) min model for U

Assume D supports ample $\quad D \in |-K_Y|$

$\Downarrow \quad$ (e.g. Y-fan)

(U is affine)

D has 0-stratum $\Rightarrow \Sigma_{(Y,D)}$ has $\dim Y = \dim$ cone

$\longleftrightarrow$ max ∂.

$NE(Y) = \{ [c] \in H_2(Y, \mathbb{Z}) \}$ is a $\overset{\text{toric}}{\text{monoid}}$ (f.g.)

$\underset{\text{a curve}}{c \subset Y}$

Let R: asso monoid ring $K[NE(Y)]$

$$R = \bigoplus_{c \in NE(Y)} K \cdot z^c \qquad (z^{c_1} \cdot z^{c_2} = z^{c_1 + c_2})$$

$\text{Spec}(K[NE(Y)]) = TV(NEF(Y)) \supset$

Conj (GHKS) Let A be free R-module with basis $U^{trop}(\mathbb{Z})$, $\theta_q \in A$

Basis elt $\longleftrightarrow q \in U^{trop}(\mathbb{Z})$

$$A = \bigoplus_{q \in U^{trop}(\mathbb{Z})} R\theta_q$$

Conj: • A has a natural R-~~module~~ alg structure

$\longleftrightarrow \quad \theta_p \cdot \theta_q = \sum \alpha(p, q, r)\, \theta_r$

$\longleftrightarrow \quad \alpha(p, q, r, \underset{NE(Y)}{\Gamma}) \in \hat{k}$ (actually integers)

$\bullet$ $\mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$
$\overset{\shortparallel}{V}$ $\qquad\qquad \overset{\shortparallel}{TV}(NEF(Y))$

$$\begin{array}{ccl} V & \longrightarrow & TV(NEF(Y)) \\ \cup & & \cup \\ V & \longrightarrow & T_{Pic(Y)} = Pic(Y) \otimes \mathbb{C}^* \end{array}$$

Flat family of affine log CY's.

(3)

(4) $\langle D_i \rangle \subset Pic(Y)$

$T^D \subset T_{Pic(Y)}$ $\leftarrow$ each $\mathcal{O}_q$ as eigen fun.

Take quotient by T^D

$V \longrightarrow T_{Pic(Y)} / T^D$ $\leftarrow$ Basis of $\mathcal{O}$s up to ______

$\qquad\qquad \overset{\shortparallel}{T_{pic(U)}}$

(5) $U^{\vee}$:= fibres

$(U^{\vee})^{\vee} = U$.

U itself is an ~~basis~~ output of such a machine

so get basis B_U of $\mathcal{O}(U) = (U^{\vee})^{trop}(\mathbb{Z})$

Th^m: True dim 2 & for clusters except dim 2

Structure constants.

$\mathcal{U}^{trop}(\mathbb{R}) \overset{\sim}{=} \mathbb{R}^N$ piecewise linear

Conj. $\exists$ distinguished imm-trees

Broken lines
tropical discs

Currently : comb^n def^n of $\longrightarrow$

Soon - GS - log curves

KY - geom def^n.

Z over F $\qquad Z/F = \mathbb{C}((t))$

$Z = \{ (z, |\ |) \mid \overset{z \in Z}{\mathbb{K}(z)} \supset F = \mathbb{C}((t))$

$\cup \qquad\qquad | \ |$ Norm on ext. dis Norm on F

$Z(F)$ take $\mathbb{K}(z) = F$

$\mathcal{U}/\mathbb{C}$, $\mathcal{U}^{an} = (\mathcal{U} \times F)$

$\mathcal{U}^{trop}(\mathbb{R}) \subset \qquad \supset \mathcal{U}(F)$

Pick (Y, D) N.C. compactification

$$\begin{array}{l} \mathcal{U}^{an} \xrightarrow{S_{YZ}} |\Sigma_{(Y,D)}| \subset \mathbb{R}^D \\ \ell \downarrow \quad \supset \mathcal{U}(F) \\ Y_c \leftarrow \end{array}$$

variety

$F^\circ = \mathbb{C}[[t]]$

 $= \mathrm{Spec}(F) \xrightarrow{f} \mathcal{U}$

$\qquad\qquad \cap \qquad\qquad \cap$

 $\mathrm{Spec}(F^\circ) \xrightarrow{f} Y$

For each $D_i \subset D$, $\# = \cap \ \# \ f(\ ) \subset Y$

For $\mathcal{U} \to Y_c$ map, $f \mapsto f(\text{closed pt})$

$$\begin{tikzcd} \mathcal{U}^{an} \arrow[r, "S"] \arrow[d, "l"'] & \Sigma(Y,D) \\ Y_e \arrow[ur, dashed, "\iota"'] & \end{tikzcd}$$

$l(S^{-1})(<\max \text{cone})$

$\text{pt} \overset{\shortparallel}{=} 0\text{-strap of } D$

l crushes gen. fib of SYZ