3/28/2016

Scattering diagrams, Broken lines & Theta functions

— Mark Gross

Fix data:
- $N \cong \mathbb{Z}^n$ Lattice
- $M = \mathrm{Hom}(N, \mathbb{Z})$ Dual lattice
  $M_{\mathbb{R}} = M \otimes \mathbb{R}$
- skew pairing $\{\ ,\ \} : N \times N \longrightarrow \mathbb{Z}$

Paring induces a map $p^* : N \longrightarrow M$
$n \longmapsto \{n, \cdot\}$
- Seed data, i.e. a basis $e_1, \dots, e_n$ of N
  $B_{ij} = \{e_i, e_j\}$

Let $N^+ = \{ \sum a_i e_i \mid a_i \geq 0,\ \sum a_i > 0 \}$

<u>Def^n</u>. A <u>wall</u> in $M_{\mathbb{R}}$ is a pair (d, f_d)
where
- d is a codim 1 rational polyhedral cone in $M_{\mathbb{R}}$ contained in $n^\perp$ for some $n \in N^+$
- $f_d = 1 + \sum_{k \geq 1} C_k z^{k p^*(n)}$

We say a wall is incoming if $p^*(n) \in d$, otherwise it is outgoing

Remark: $p^*(n) \in n^\perp$ since $\langle p^*(n), n\rangle = \{n, n\}$
(as pairing is skew) $\leftarrow$ $= 0$

Assume p^* is injective (B is invertible)
$\begin{pmatrix} B & I \\ -I & 0 \end{pmatrix}$ will always work for principal coeff.

Then choose a strictly convex rational polyhedral cone $\sigma \subseteq M_{\mathbb{R}}$ such that
$p^*(N^+) \subseteq \sigma$

Set $P = \sigma \cap M$ (monoid)

$k[P] = \bigoplus_{p \in P} k z^p$, $z^p \cdot z^q = z^{p+q}$ (monoid ring)

Let $m = \langle z^p \mid p \in P \setminus \{0\} \rangle \subseteq k[P]$ be the max. ~~ideal~~ monomial ideal

$\widehat{k[P]} = \varprojlim k[P]/m^k$, the completion of $k[P]$ w.r.t m.

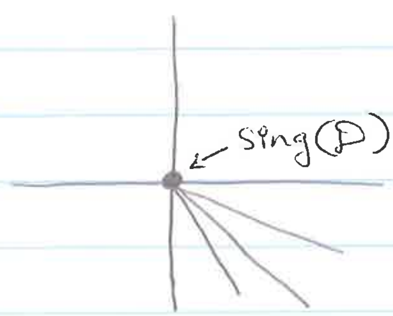
So $f_d \in \widehat{k[P]}$

Def^n: A scattering diagram $\mathfrak{D}$ is a set of walls s.t. $\mathfrak{D}_k = \{ (d, f_d) \in \mathfrak{D} \mid f_d \not\equiv 1 \mod m^{k+1} \}$ is finite $\forall k$.

Def^n: $\mathrm{Supp}(\mathfrak{D}) = \bigcup_{d \in \mathfrak{D}} d \subseteq M_{\mathbb{R}}$

$\mathrm{Sing}(\mathfrak{D})$ = "locus where $\mathrm{Supp}(\mathfrak{D})$ is not a manifold"

$$= \bigcup_{\substack{d_1, d_2 \in \mathfrak{D} \\ \dim d_1 \cap d_2 \leq n-2}} d_1 \cap d_2 \ \cup \ \bigcup_{d \in \mathfrak{D}} \partial d$$



Def^n: Let $\gamma : [0,1] \longrightarrow M_{\mathbb{R}} \setminus \mathrm{Sing}\, \mathfrak{D}$ be a differentiable path f $\gamma(0), \gamma(1) \in M_{\mathbb{R}} \setminus \mathrm{Supp}\, \mathfrak{D}$ crossing walls transversally.

Will define the path order product

$$\theta_{\gamma, \mathfrak{D}} : k[\widehat{P}] \longrightarrow k[\widehat{P}] \quad \text{a } k\text{-alg hom}$$

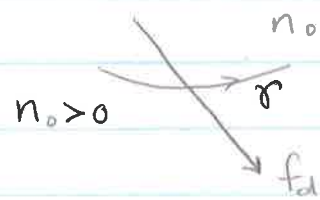
If $(d, f_d) \in \mathfrak{D}$ with $\gamma(t_0) \in d$, define

$\theta_{\gamma, d} : k[\widehat{P}] \circlearrowleft$ by

$$\theta_{\gamma,d}(z^p) = z^p f_d^{\langle n_0, p \rangle} \quad \text{where}$$

$n_0 \in N$ is such that

- $d \subseteq n_0^{\perp}$, n_0 primitive, $\langle \gamma'(t_0), n_0 \rangle < 0$



Then define

$$\theta_{\gamma, \mathfrak{D}_k} = \theta_{\gamma, d_r} \circ \cdots \circ \theta_{\gamma, d_1}$$

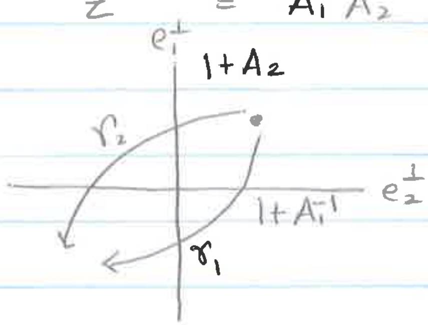
where $d_1, \dots, d_r \in \mathfrak{D}_k$ are the walls crossed by γ in the order crossed.

$$\theta_{\gamma, \mathfrak{D}} = \lim_{k \to \infty} \theta_{\gamma, \mathfrak{D}_k}$$

Example: $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad p^*(e_1) = (0,1)$, $\quad p^*(e_2) = (-1,0)$

$$k[\widehat{P}] = k[[z^{(0,1)}, z^{(-1,0)}]]$$

$$z^{(a,b)} = A_1^a A_2^b$$



$= z^{(1,0)}$

$\theta_{\gamma, \mathfrak{D}} : A_1 \longmapsto A_1 \longmapsto A_1(1+A_2)^{1}$

$A_2 \longmapsto A_2(1+A_1^{-1}) \longmapsto$

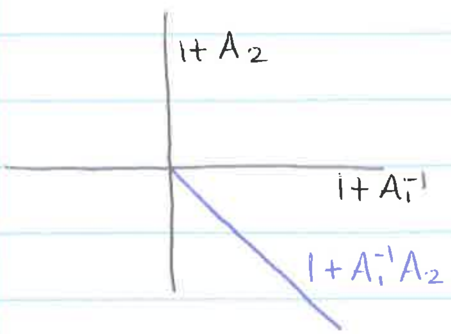
$\longrightarrow A_2(1+A_1^{-1}(1+A_2)^{-1})$

$$\gamma_2: \quad A_1 \xmapsto[\text{wall}]{\text{vert.}} A_1(1+A_2) \xmapsto[\text{wall}]{\text{hori}} A_1(1+A_2(1+A_1^{-1}))$$
$$A_2 \longmapsto A_2 \longmapsto A_2(1+A_1^{-1})$$

Def^n: A scattering diag $\mathfrak{D}$ is said to be consistent if $\Theta_{\gamma,\mathfrak{D}}$ only depends on the endpoints of γ.

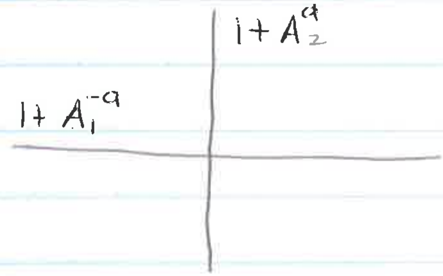
(the scattering diag in the prev. ex. is not consistent)

Th^m: Let $\mathfrak{D}_{in}$ be a scattering diag. Then $\exists! \mathfrak{D} \supseteq \mathfrak{D}_{in}$ s.t.
- $\mathfrak{D}$ is consisten
- $\mathfrak{D} \setminus \mathfrak{D}_{in}$ consists only of outgoing walls



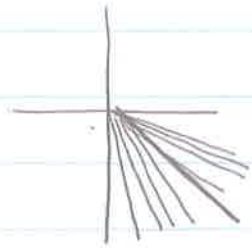
Proof: 2 dim - Kontsevich + Sorbelman 2004
all dim - Gross, Siebert 2007

Ex. $B = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ $\mathfrak{D}_{in}$

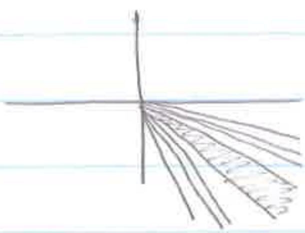


For $a = 2$

$$\mathfrak{D} = \mathfrak{D}_{in} \cup \{(\mathbb{R}_{\geq 0}(n, -(n+1)), 1 + A_1^{2n} A_2^{2n+2}) \mid n \geq 1\}$$
$$\cup \left\{\left(\mathbb{R}_{\geq 0}(1,-1), \frac{1}{(1 - A_1^{-2} A_2^{2})^4}\right)\right\}$$



For $a \geq 3$



The cluster Scattering diagram

Take $\mathfrak{D}_{in} = \{(e_i^{\perp}, 1 + z^{p^*(e_i)}) \mid 1 \leq i \leq n\}$

$\mathfrak{D}$ is the associated consistent scattering diagram.

Thm: For this scattering diagram, all f_d have positive integer coefficients.

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Speaker's Name: Mark Gross

Talk Title: Scattering diagrams, broken lines & theta functions

Date: 3 / 29 / 2016 **Time:** 11 : 30 am / pm (circle one)

List 6-12 key words for the talk: scattering diagrams, broken lines, theta functions

Please summarize the lecture in 5 or fewer sentences:

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3/29/2016

Scattering diagrams, broken lines, theta functions
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We have $\{\ ,\ \} : N \times N \longrightarrow \mathbb{Z}$

$e_1, \dots, e_n$ basis for N

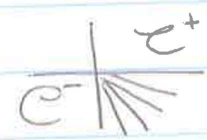
$\mathfrak{D}_{in} = \{ (e_i^\perp, 1 + z^{p^*(e_i)}) \mid 1 \le i \le n \}$

$\longrightarrow \mathfrak{D}$ is consistent

Remark: Let $\mathcal{C}^+ = \{ m \in M_{\mathbb{R}} \mid \langle m, n \rangle > 0 \ \forall n \in N^+ \}$

Then $\mathrm{Supp}(\mathfrak{D}) \cap \mathcal{C}^+ = \phi$

Similarly for $\mathcal{C}^-$



Recall mutation of the seed means choosing a new basis $\mu_k(e_1, \dots, e_n) = (e_1', \dots, e_n')$

with $e_i' = \begin{cases} e_i + \max(B_{ik}, 0)\, e_k & i \neq k \\ -e_k & i = k \end{cases}$

This gives a new scattering diagram $\mathfrak{D}'$

Q: What's the relationship with $\mathfrak{D}$?

Let $\mathcal{H}_{k,\pm} = \{ m \in M_{\mathbb{R}} \mid \langle e_k, m \rangle \ {>0 \atop <0} \}$

Define $T_k : M_{\mathbb{R}} \to M_{\mathbb{R}}$ by

$T_k(m) = \begin{cases} m + p^*(e_k) \langle e_k, m \rangle, & m \in \mathcal{H}_{k,+} \\ m & m \in \mathcal{H}_{k,-} \end{cases}$

How does T_k act on $\mathfrak{d}$?

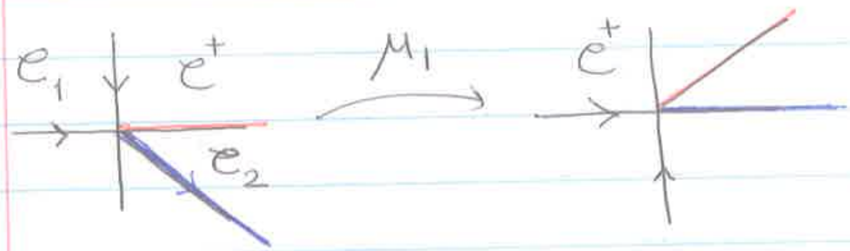
For a wall $(\mathfrak{d}, f_{\mathfrak{d}})$, assume either $\mathfrak{d} \subseteq \mathcal{H}_{k,+}$ or $\mathfrak{d} \subseteq \mathcal{H}_{k,-}$

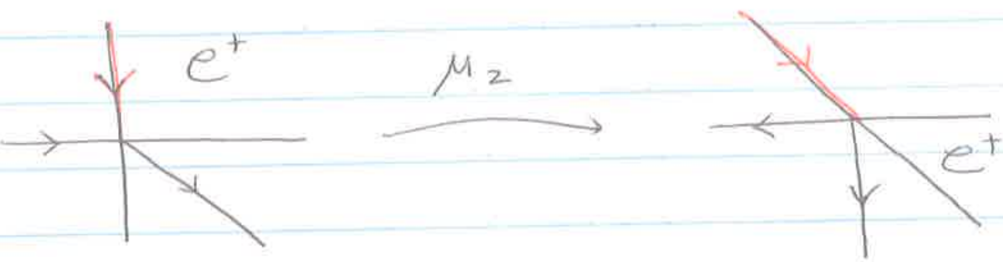
$\longrightarrow$

Then define $T_k(d, f_d) = (T_k(d), T_k(f_d))$
where $T_k(f_d)$ is defined by applying linear transformation $T_k|_{\mathcal{H}_{k,\pm}}$ to the exponents of f_d.

Define $T_k(\mathfrak{D}) = \{ T_k(d, f_d) \mid (d, f_d) \in \mathfrak{D} \setminus \{(e_k^\perp, 1+z^{p^*(e_k)})\} \}$
$\cup \{ (e_k^\perp, 1 + z^{-p^*(e_k)}) \}$.

Th^m: $\mathfrak{D}' = T_k(\mathfrak{D})$





By pulling back $\mathcal{C}^+$ under the mutations / the maps T_k, (labelled $\mathcal{C}_1, \mathcal{C}_2$ in the diag)
get a map
$$\left\{ \begin{matrix} \text{seeds obtained by} \\ \text{mutation from } e_1, \dots, e_n \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{connected comp. of} \\ M_{\mathbb{R}} \setminus \operatorname{Supp} \mathfrak{D} \end{matrix} \right\}$$

Denote by Δ^+ the image of this map.
Given $\mathcal{C}_1, \mathcal{C}_2 \in \Delta^+$ adjacent, the wall-crossing automorphism is always of the form $z^m \longmapsto z^m(1 + z^{p^*(n_0)})^{\langle m, n_0 \rangle}$
This defines a map $\operatorname{Quot}(k[M])$ to itself

inducing a birational map

$\Psi_{e_1,e_2} : \mathrm{Spec}\, k[M] \longrightarrow \mathrm{Spec}\, k[M]$

If we associate a torus $T_{M,e} = \mathrm{Spec}\, k[M]$ for any $e \in \Delta^+$, we can glue together all these tori via these birational maps. This is possible by consistency of $\mathfrak{D}$.

Thm: This is the $\mathcal{A}$ Cluster variety associated to the initial seed.

Cor: A regular function on $\mathcal{A}$ is given by a Laurent polynomial $f \in k[M]$ such that $\theta_{\gamma,\mathfrak{D}}(f)$ is a Laurent polynomial $\forall$ paths γ joining e^+ to $e \in \Delta^+$

Note: Such an f is called a universal Laurent polynomial.

Ex. $z^{e_i^*}$ is a univ. Laurent poly.

(the Laurent phenomenon)

Broken Lines:

Def^n A broken line for $m_0 \in M \setminus \{0\}$ and endpoint $Q \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$ is a piecewise linear path $\gamma : (-\infty, 0] \longrightarrow M_{\mathbb{R}}$ with a finite # of domains of linearity & for each domain of linearity L, a monomial $c_L z^{m_L} \in k[M]$ such that $\longrightarrow$

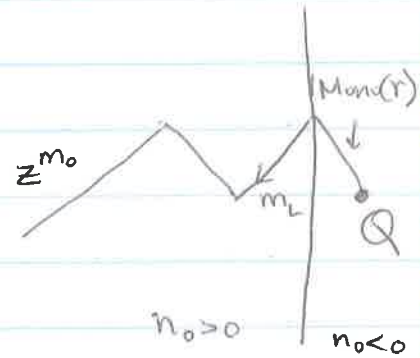
① $\gamma'(t) = -m_L$ for $t \in L$

② the initial monomial is z^{m_0}

③ $\gamma(0) = Q$

④ γ bends only when it crosses a wall $(\mathfrak{d}, f_{\mathfrak{d}})$, passing from domain L', if $c_{L'} z^{m_{L'}}$ is a term in

$$\theta_{\gamma,\mathfrak{d}}(c_L z^{m_L}) = c_L z^{m_L} f_{\mathfrak{d}}^{\langle m_L, n_0 \rangle}$$



Remark : The exponent of $f_{\mathfrak{d}}$ is always positive.

Def^n If γ is a broken line, write Mono(γ) for the last attached monomial. Define

$$\vartheta_{Q,m_0} = \sum_{\gamma} \mathrm{Mono}(\gamma) \in z^{m_0} k[P]$$

where the sum is over all broken lines for m_0 with endpoint Q.

Th^m: If γ is a path joining Q and Q', then $\vartheta_{Q',m_0} = \theta_{\gamma,\mathfrak{D}}(\vartheta_{Q,m_0})$

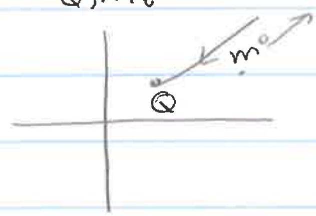
eg. If ϑ_{Q,m_0} is a (Laurent) polynomial for every $Q \in \mathcal{C} \in \Delta^+$, then ϑ_{Q,m_0} is a univ. Laurent polynomial for $Q \in \mathcal{C}^+$

Facts: ① Coefficients of ϑ_{Q,m_0} are manifestly positive.

② If ϑ_{Q,m_0} is a polynomial for Q in some $\mathcal{C} \in \Delta^+$, then it's a polynomial for Q in all $\mathcal{C} \in \Delta^+$

(3) If $m_0 \in \overline{\mathcal{C}^+}$, $Q \in \mathcal{C}^+$, then $\vartheta_{Q,m_0} = z^{m_0}$

(Once it leaves the positive chamber, it never comes back to it)



Cor: Positivity of the Laurent phenomenon.

Q: Which $m_0 \in M$ give univ. Laurent polynomials?

Let $\Theta = \{ m_0 \in M \mid \vartheta_{Q,m_0}$ is a univ. Laurent poly $\}$ $\cup \{0\}$
$Q \in \mathcal{C}^+$

Facts:
- Θ is closed under addition
- Θ is saturated i.e. if $rm \in \Theta$ for $r > 0$ then $m \in \Theta$

e.g. $n = 2$, $\Theta = M$

- $\left(\bigcup_{\mathcal{C} \in \Delta^+} \overline{\mathcal{C}} \right) \cap M \subseteq \Theta$