Examples of cluster varieties & their scattering diags
- Paul Hacking

Geometric description of cluster varieties as blowup of toric varieties

$(\bar{Y}, \bar{D})$ toric variety + toric boundary

$\bar{Y} \setminus \bar{D} = T \simeq (\mathbb{C}^*)^n$

$\bar{\sigma}$ hol. 2-form on T, log poles along $\bar{D}$

$$\Rightarrow \quad \bar{\sigma} = \sum \underset{\substack{\cap \\ \mathbb{C}}}{a_{ij}} \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$$

$\pi : Y \to \bar{Y}$

sequence of blowups of subvarieties $Z \subset \bar{Y}$

s.t. $\bar{\sigma}$ lifts to σ hol form on $Y \setminus D$ (*)

$D = \bar{D}'$ strict transform

In dim 2:

$U = Y \setminus D$

$E \not\subset D$!

$U = Y \setminus D$ is cluster variety

(*) very restrictive

Must have $Z = C \cap \overline{(X = \lambda)}$

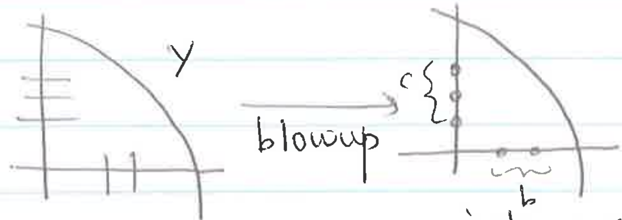
$C \subset \bar{D}$ cpt of $\bar{D}$, $X : T \to \mathbb{C}^\times$ character

$\lambda \in \mathbb{C}^\times$

further $\operatorname{Res} \bar{\sigma}|_C = v \cdot \frac{dx}{x}$, $v \in \mathbb{C}^\times$

Ex. $\mathcal{A}(b,c) = H^0(\mathcal{O}_U)$

$U = Y \setminus D$



$\bar{Y} = \mathbb{P}^2$

points are $x^b + 1 = y = 0$

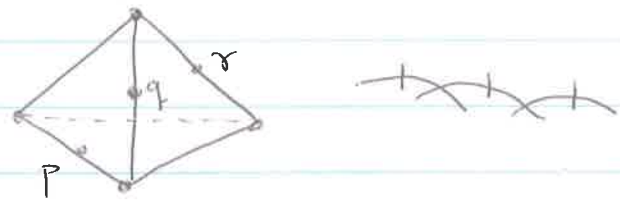
$y^c + 1 = x = 0$

Ex 2 Cluster variety for A_n.

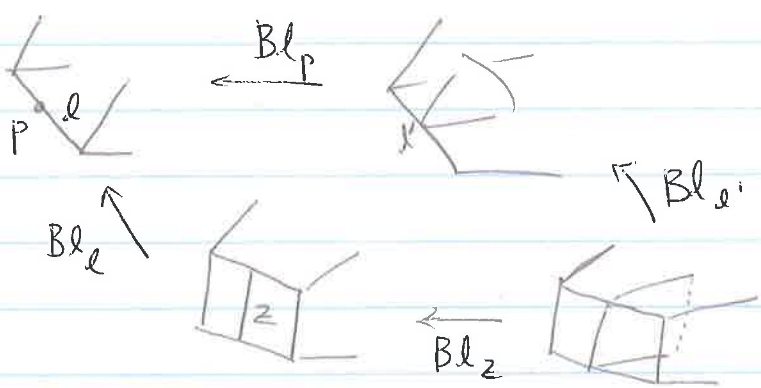
($\mathcal{X}$-variety ($\simeq \mathcal{A}$-variety if n even))

$Y \xrightarrow{\text{blowup}} \bar{Y} = \mathbb{P}^n$

$X = U = Y \setminus D$



To fit into previous description, first blowup lines



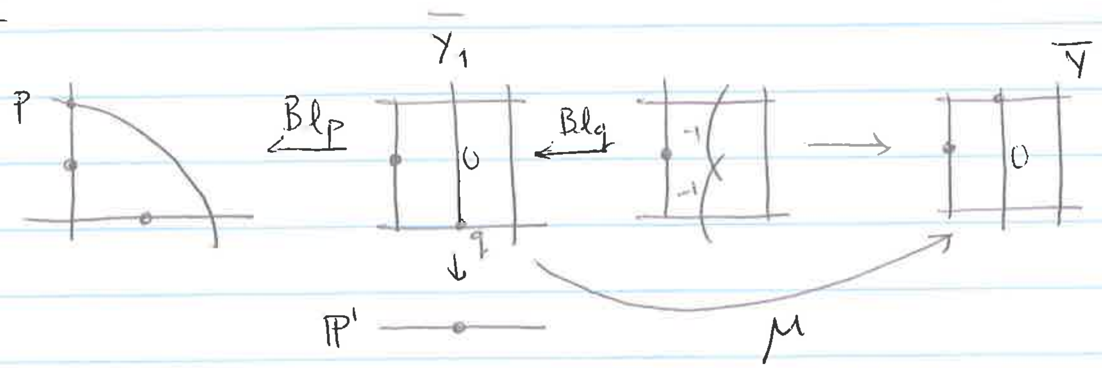
Mutation corresponds to elementary transformations of $\mathbb{P}^1$-bundles

$$(Y, D) \xrightarrow{\pi} (\bar{Y}, \bar{D}) \qquad \mu : T \dashrightarrow T' \text{ (tori)}$$
$$U \supset \bar{U} = T \qquad\qquad \begin{array}{ccc} T & \dashrightarrow & T' \\ \cap & & \\ \bar{Y} & \dashrightarrow & \bar{Y}' \end{array}$$

(birat'l trans)

dim 2

$\mathcal{A}(1,1)$



Mirror symmetry:

Cluster varieties come in mirror pairs U, V

s.t. sympl. geom of $U \longleftrightarrow$

If skew symm (or rank 2) $\quad U, V = \mathcal{X}, \mathcal{A}$

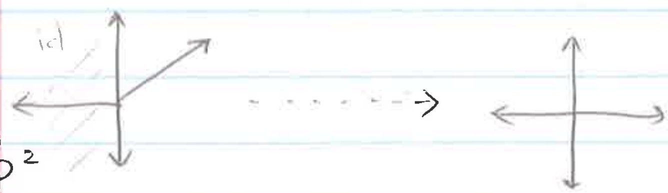
$$H^0(V, \mathcal{O}_V) = \bigoplus_{q \in U^{trop}(\mathbb{Z})} \mathbb{C}\, \theta_q$$

$$U^{trop}(\mathbb{Z}) = \{(E, M) \mid E \text{ boundary div in some cpct of } U \text{ s.t. } \Omega \text{ has pole along } E,\ M \in \mathbb{N}\}$$

$$\cup \{0\}$$
$$\updownarrow$$
$$\theta_0 = 1$$

Scattering diagram $\mathfrak{D}$: drawn in $U^{trop}(\mathbb{Z})$
encodes :
1. counts of hol discs in U
2. Gluing clusters tori of V

$U^{trop}(\mathbb{Z})$

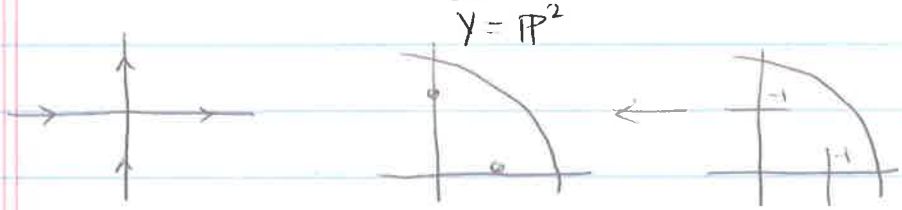


$Bl_p \mathbb{P}^2$ $\qquad$ $\mathbb{P}^1 \times \mathbb{P}^1$

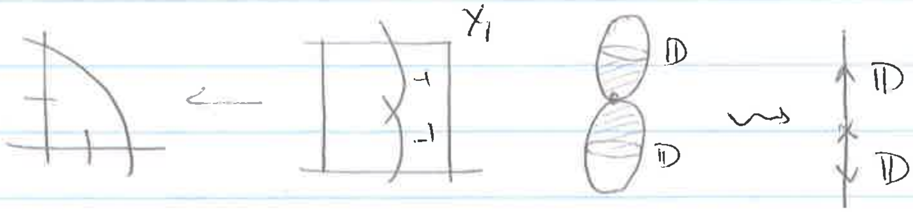
$U^{trop}(\mathbb{Z})$ is identified in piecewise linear way
$N = \mathrm{Hom}(\mathbb{C}^\times, T) = H_1(T, \mathbb{Z})$ for any $T \subset U$

Enumerative $\qquad$ Fukaya 2001

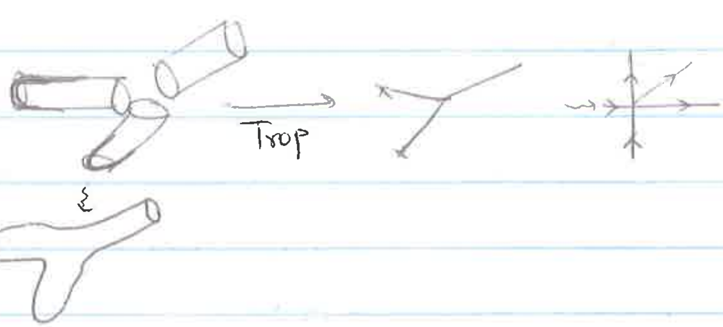
$\bar{Y} = \mathbb{P}^2$



X_1



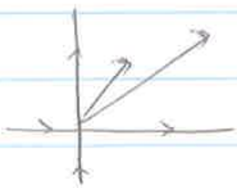
Our diag.
Pushing to ∞



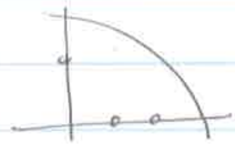
Trop

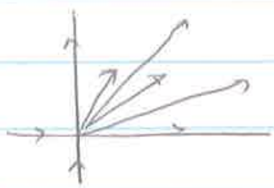
Same for

L : line $\overline{pq}$

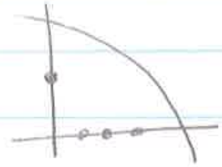


B_2





G_2



Ex. Scat. diag. for acyclic seed for a cluster variety.

$e_1, \dots, e_n \in N \simeq \mathbb{Z}^n$

$\{\ ,\ \} : N \times N \longrightarrow \mathbb{Z}$ skew

seed data for $\mathcal{A}$ variety

$\rightsquigarrow$ Quiver Q vertices $1, 2, \dots, n$

$\#$ arrows from i to j $:= \max(\{e_i, e_j\}, 0)$

Say seed is acyclic if $\nexists$ directed cycle

Ex. Finite type ($\#$ clusters $< \infty$) $\Longleftrightarrow$ in some seed Q is a Dynkin diag (with some orientation) in particular, acyclic.

Reineke used reps of quivers to study scattering diag in this case 0804.3214

$N = \mathbb{Z}^n$ basis $e_1, \ldots, e_n$

$M = N^* = \mathbb{Z}^n$ with dual basis

$x \in M_{\mathbb{R}}$ defines a notion of stability for quiver rep.

Say rep V is x-stable if $\langle x, d \rangle = 0$

$d = (\dim V_i) \in N = \mathbb{Z}^n$ & for any $V' \subset V$,

$\langle x, d' \rangle = 0$

(maybe call "stable of slope zero")

$\rightsquigarrow$ moduli spaces of stable reps of Q

Thm (GHKK Prop 8.28 $\Leftarrow$ Reineke '08)

Let $x \in \cancel{\ } M_{\mathbb{R}}$ & assume $\exists!\ d \in (\mathbb{Z}_{\geq 0})^n \subset \mathbb{Z}^n = N$ s.t. $\langle x, d \rangle = 0$. Then (1) if $\exists$ wall ∂ of $\mathfrak{D}$ through x $\Rightarrow$ $\exists$ x-stable rep of Q with $\dim(V) = k \cdot d$ for some k

& attached function $f = \left(1 + \sum_{k \geq 1} e(\mathcal{M}^x_{kd,i}(Q)) \cdot z^{k\{d,\cdot\}}\right)^{1/d}$

where $\mathcal{M}^x_{kd,i}(Q)$ = moduli of reps of Q, x-stable, $\dim = kd$, "framed" at vertex i"

In the Dynkin case (ADE), the indecomposable reps of Q are rigid (moduli = pt) with dim vector $d \in \Phi^+$ positive roots (Gabriel's thm)

$\Rightarrow f = 1 + z^{\{d,\cdot\}}$, can determine a wall through x in $d^\perp$ by analyzing stability.