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Spectral Networks and Noncommutative Cluster Algebras

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Data : • $G = U(K)$ $\quad G_{\mathbb{C}} = GL(K, \mathbb{C})$

• Riem. Surface C with $n \geq 0$ punctures z_i

• $m_i \in \mathfrak{g}_{\mathbb{C}}$, $m_i^{\mathbb{R}} \in \mathfrak{g}$ ~~semisimple~~ diagonal

• rank K Herm. vector bundle E over C

Hitchin system : [Hitchin, Simpson]

$$\mathcal{M} = \mathcal{M}(G, C) = \left\{ \begin{array}{l|l} \text{unitary conn. } D \text{ in } E & F_D + [\varphi, \varphi^\dagger] = 0 \\ \varphi \in \Omega^{1,0}(\text{End } E) & \bar\partial_D \varphi = 0 \\ \text{w/ poles at } z_i & \\ \text{residues } m_i^{\mathbb{R}}, m_i & \end{array} \right\} \Big/ \sim$$

Facts about $\mathcal{M}$:

1) $\mathcal{M}$ is hyperkähler, $\Rightarrow$ has kähler structure $(\mathcal{M}, I_\zeta, \omega_\zeta)$ $\zeta \in \mathbb{CP}^1$

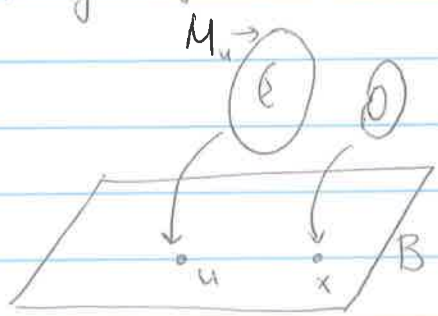
$\Downarrow$

CY $\qquad$ hol. sympl. form Ω_ζ

2)
$$\begin{array}{ccc} \mathcal{M} & & (D, \varphi) \\ \downarrow \pi & & \downarrow \\ B & & \Sigma_u = \{ \det(\varphi - \lambda) = 0 \} \subset T^*C \end{array}$$

B = lin. sys of $K[C]$ in T^*C

$\mathcal{M}_u = \text{Pic}^d \, \Sigma_u$ (generic u)



bad fibers : where Σ_u singular / not reduced

$|\zeta| = 1 \Rightarrow$ this is special Lag. fibration

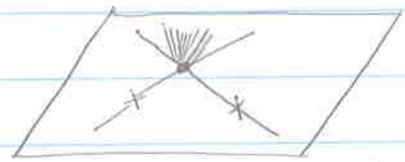
$\zeta = 0 \Rightarrow$ this is hol fibration

3) For $\zeta \in \mathbb{C}^\times$,

$(\mathcal{M}, I_\zeta, \Omega_\zeta) \simeq \mathcal{M}^b(GL(K, \mathbb{C}), C)$

$\left(\nabla(\zeta) = \frac{\varphi}{\zeta} + D + \varphi^\dagger \zeta \right)$

Scattering diagram



Fix $\zeta \in \mathbb{C}^\times$ $|\zeta| = 1$

Scatt diag D on B

Moral def^n of D:

$u \in B : \longrightarrow \Sigma_u \subset T^*C$

(nbhd of C in) T^*C is hyperkähler

$\Sigma \subset T^*C$

$D = \{ u \in B \mid \exists\ I_\zeta\text{-hol disc in } T^*C \text{ with } \partial \text{ on } \Sigma_u \}$

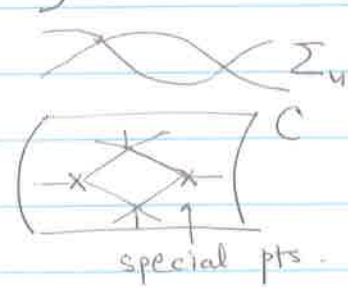
actual def: tropical version of this

Canonical coordinate

Conj: Given $u \in B \setminus D$, $\exists$ canonical hol. Darb coord. system on $U \ni \pi^{-1}(u)$

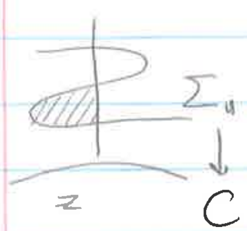
$\Psi_u : U \longrightarrow T_u = H^1(\Sigma_u, \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^{2n}$

characterized by following



Build a network $W(u) \subset C$

$W(u) = \{ z \in C \mid \exists\ I_\zeta\text{-hol. bigons in } T^*C \text{ w/ } \partial \text{ on } \Sigma_u \ \&\ T^*_z C \}$

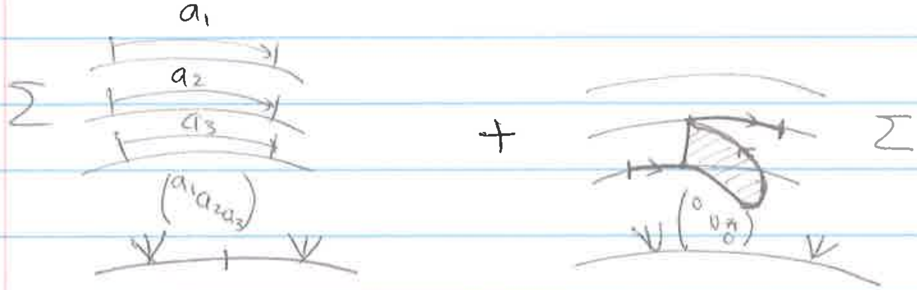


$$\underset{\text{local}}{\Psi_u} : \mathcal{M}^b(GL(K,\mathbb{C}), C) \xrightarrow{\sim} \mathcal{M}^b(GL(1,\mathbb{C}), \Sigma_u)$$

$$\nabla \longmapsto \nabla^{ab}_u \qquad \text{(ab: abelian)}$$

s.t. $\nabla \simeq \pi_* \nabla^{ab}_u$ on $C \setminus W(u)$

$(\nabla\text{-transport}) = (\pi_*\nabla\text{-transport}) + \left(\begin{matrix}\text{corr. from } I_\zeta\text{-hol.}\\ \text{bigons}\end{matrix}\right)$



More fundamentally:

$$F_u : \mathbb{Z}[\pi_{\le 1} C] \longrightarrow \mathbb{Z}[\pi_{\le 1}\Sigma]$$

$$\nabla\text{-transport}(p) = \nabla^{ab}\text{ transport}(F_u(p))$$

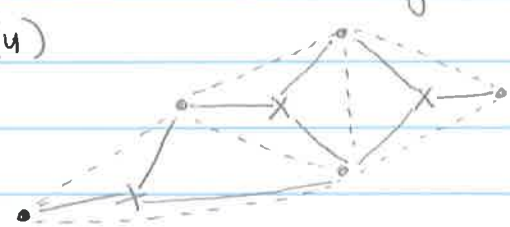
Consequences:

1) For $p \subset C$ closed curve:

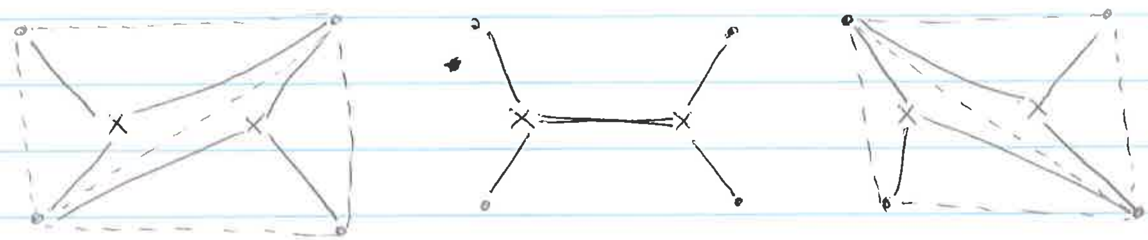
$$\underset{\substack{\uparrow\\ \Theta\text{-function}}}{\mathrm{Tr}\,(\mathrm{Hol}_p \nabla)} = \sum_{\gamma \in H_1(\Sigma,\mathbb{Z})} \underset{\substack{\uparrow\\ \text{Counting}\\ \text{broken lines}}}{\overline{\Omega}_u(p,\gamma)}\ \underbrace{\mathrm{Hol}_\gamma \nabla^{ab}_u}_{\substack{\shortparallel\\ X_{\gamma,u}}}$$

(for simple closed curves)

- When $k=2$, and have $n \geq 1$ punctures, the $X_{\gamma,u}$ are Fock-Goncharov cluster $\mathcal{X}$ coordinates attached to triangulation $T(u)$



• When u crosses D, topology of $W(u)$ changes:



$F_u : \mathbb{Z}[\pi_{\le 1} C] \longrightarrow \mathbb{Z}[\pi_{\le 1}\Sigma] \circlearrowleft K$

F_u changes by post-comp with K

Ex.

pass to the subquotient $\mathbb{Z}[H_1\Sigma]$, K descends to cluster $\mathcal{X}$ move

$X_\gamma \longrightarrow X_\gamma(1+X_\mu)^{\langle\mu,\gamma\rangle}$