Cluster duality and mirror symmetry for the Grassmannian
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Joint work w/ Konstanze Rietsch

A-model variety X B-model variety $X^\vee$ with superpotential

Def^n Grassmannian $Gr_k(\mathbb{C}^n) = \{V \subset \mathbb{C}^n \mid \dim V = k\}$

represents elmts by $k \times n$ matrices M

For $I \in \binom{[n]}{k}$, $P_I(M) =$ det of $k \times k$ submatrix of M in columns I

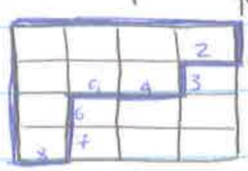
Overview

A-model   B-model

$X = Gr_{n-k}(\mathbb{C}^n)$ $X^\vee =$ complement of anti-canon divisor in $Gr_k(\mathbb{C}^n)$

- Remove locus where any of cyc. consec. Pluckers vanish

partition $\leftrightarrow$ vert steps $N = k(n-k)$



partition $\leftrightarrow$ horiz. steps.

This is how we index Plucker coord's.

Fix red. plabic graph G w/ perm $\pi_{k,n}$

"plabic" chart   cluster chart

$\Phi_G : (\mathbb{C}^*)^N \to X$ $\Phi_G^\vee : (\mathbb{C}^*)^N \to \check{X}$

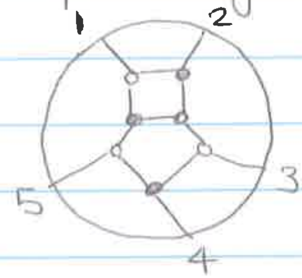
$\mathcal{X}$-variety $\mathcal{A}$-variety

Newton-Okounkov body   Write superpotential $W : \check{X} \to \mathbb{C}(q)$ in terms of cluster of G and "tropicalize it"

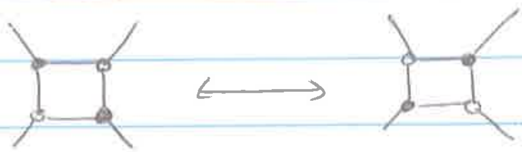
NO_G: $\frac{\text{NObody}}{\text{arb. variety}} = \frac{\text{moment poly}}{\text{toric variety}}$ polytope Q_G

Th^m: (Rietsch-Williams) $NO_G = Q_G$

Def^n (Postnikov): Plabic graph G is a planar graph ~~d~~ in a disk with n boundary vertices. Each bdry vertex has deg 1. Internal vertices black or white.



Moves on G

(M1) 

(M2) 

(M3)  (similar for hollow vertex)

Def^n: G is reduced if $\nexists\ G' \sim G$ s.t. G' contains local config 

Def/Lemma: "Rules of road"

Turn right at •, left at ∘. Given red. G, assoc. trip term π_G by starting at each bdry vertex i & following rules until reach another bdry vertex j. Defines permutation π_G s.t. $\pi_G(i) := j$

( take the rightmost path)

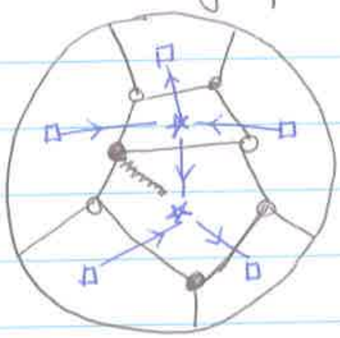
In the example above π_G : $\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & & & \\ 3 & 4 & 5 & 1 & 2 \end{matrix}$

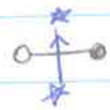
Given a Plabic graph G with $\Pi_G = \Pi_{k,n} = \begin{array}{cccccc} 1 & \cdots & k & k+1 & \cdots & n \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ n-k+1 & & n & 1 & & n-k \end{array}$

Postnikov — Scott

plabic chart (Cluster X-variety) — Cluster chart (A-variety)

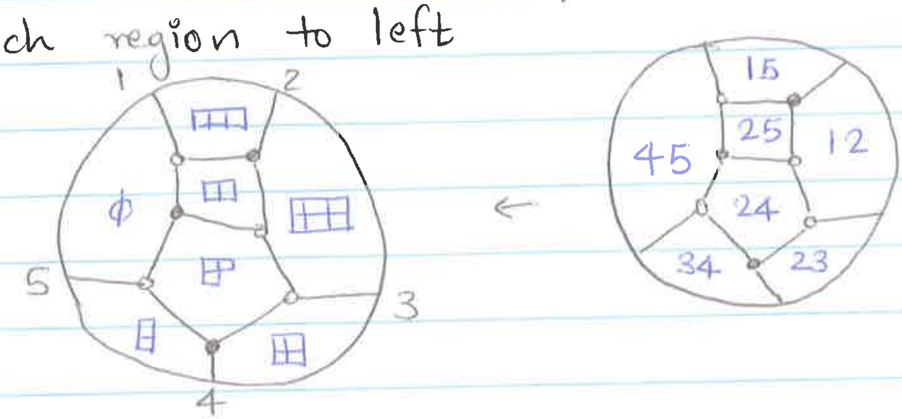
Quiver assoc. to G is dual graph



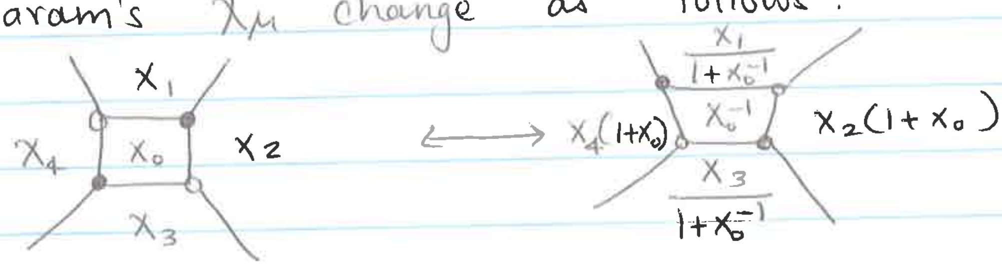


□ : frozen

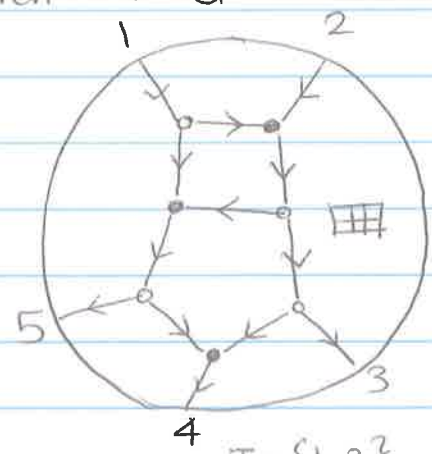
Label faces of G by partitions. The trip starting at i partitions the disk into 2 parts: left, right. Put i in each region to left



A-model: Put variable X_μ in region labeled μ in G.
Cluster X-variety: when G changes by (M1) param's X_μ change as follows:



To define Φ_G, choose a perf. orient of G
(every $\bullet$ has ! outgoing edge)
(every $\circ$ has ! ingoing edge)



$I = \{1, 2\}$

Let I = set of sources of $\mathcal{O}$

$\Phi_G : (\mathbb{C}^*)^N \longrightarrow Gr_{n-k}(\mathbb{C}^n)$

$\{x_\mu\} \longmapsto \Phi_G(\{x_\mu\})$ defined by

Plucker coord P_J = gen fun for flows (set of NI paths) I to J
non-intersecting

Plucker coords $\xleftrightarrow{\text{val}}$ Integer lattice pts in $\mathbb{R}^N$ obtained by choosing leading (lowest deg) term in P_J

$P_{\{1,2,\dots,n-k\}}$

	$(2,2)$	$(3,3)$	$(1,1,1)$	(2)	(3)	$(2,1)$	$\emptyset$
$P_{12} = 1$	0	0	0	0	0	0	0
only one path $2 \to 3$ — $P_{13} = x_{(3,3)}$	0	1	0	0	0	0	0
two paths $2 \to 4$ — $P_{14} = x_{(2,2)} x_{(3,3)} (1 + x_{(2,1)})$	1	1	0	0	0	0	0

$\vdots$

Convex hull gives polytope NO_G

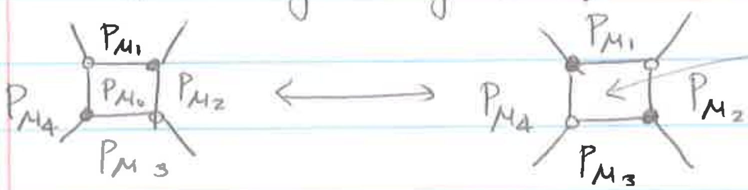
B-model : Cluster chart

$\Phi_G^\vee : (\mathbb{C}^*)^N \longrightarrow X^\vee \subset Gr_k(\mathbb{C}^n)$ ($\mathcal{A}$-variety)

Use same G with faces labeled by μ as before.

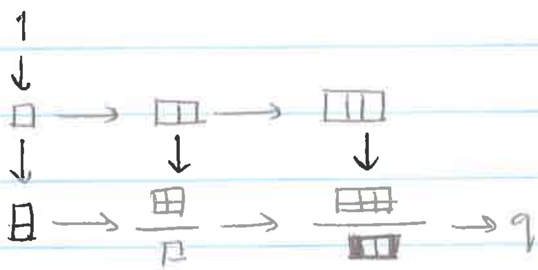
$\Phi_G^\vee : \{p_\mu\} \longmapsto \Phi_G^\vee(\{p_\mu\})$ given by expressing each Plucker coord on $Gr_k(\mathbb{C}^n)$ as Laurent poly in $\{p_\mu\}$.

If G changes by (M1)



$$\frac{P_{\mu_1}P_{\mu_3} + P_{\mu_2}P_{\mu_4}}{P_{\mu_0}}$$

On B-model side we have a super potential.
$W : \check{X} \to \mathbb{C}(q)$
Expression for W in Gr given by Marsh-Rietsch



For $Gr_3(\mathbb{C}^5)$

$$W = P_{(1)} + \frac{P_{(1,1)}}{P_{(1)}} + \frac{P_{(2)}}{P_{(1)}} + \frac{P_{(2,2)}}{P_{(1)}P_{(1,1)}} + \cdots + q\frac{P_{(3)}}{P_{(3,3)}}$$

Given cluster $\mathcal{C}_G$, associated to G,
rewrite W in terms of $\mathcal{C}_G$
In our ex of G, need to use $P_{(1)} = \dfrac{P_{(2,2)}P_{\phi} + P_{(1,1)}P_{(3,1)}}{P_{(2,1)}}$

$$W = \frac{P_{(2,2)}}{P_{(2,1)}} + \frac{P_{(1,1)}P_{(3)}}{P_{(3,2)}P_{\phi}} + \cdots$$

"tropicalize W"
Define polytope Q_G by inequalities, one for each summand of W:
$b_{(2,2)} - b_{(2,1)} \geq 0, \quad b_{(1,1)} + b_{(3)} - b_{(2,1)} \geq 0, \ldots$

Thm: $NO_G = Q_G$ $\square$