Mirror symmetry for homogeneous spaces

– Clelia Pech

1. Grassmannians [Marsh - Rietsch] [Rietsch - Williams]
2. Quadrics [Pech - Rietsch - Williams]
3. Lagrangian Grassmannians [Pech - Rietsch]

$X = G/P$ $\qquad\qquad$ $(X^\vee, W)$ LG-model

$$\begin{tikzcd} & X^\vee = \pi(R^\vee_{w_0, w_P}) \arrow[dl, "W"'] \arrow[d, phantom, "\cap"] \\ \mathbb{C} & G^\vee/P^\vee \end{tikzcd}$$

proj. open Richardson var.

ⓐ $QH^*(X)$ $\longleftrightarrow$ $\mathrm{Jac}(\check{X}, W)$ Jacobi ring

quantum cohom

ⓑ $(H^*(X,\mathbb{C}), \nabla^A)$ $\longleftrightarrow$ $(H^N_{dR}(X^\vee, d + dW_q \wedge \_), \nabla^B)$

A-model $\qquad\qquad$ B-model

(1) $Gr(k,n)$

$X = Gr(k,n)$ $\qquad\qquad$ $X^\vee \subseteq Gr(n-k, n) \subseteq \mathbb{P}(\Lambda^k \mathbb{C}^n)$

Plücker embedding

$H^*(X) = \bigoplus_{\substack{\lambda : \text{part}^{\text{ns}} \\ \subseteq k \times (n-k) \text{ rect.}}} \mathbb{C}\sigma^\lambda \xleftrightarrow{1-1} P_\lambda, \ \lambda \subseteq k \times (n-k)$

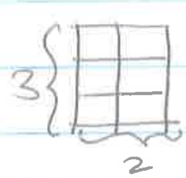
Plücker coord

Superpotential: $X = Gr(3,5) \rightsquigarrow X^\vee \subseteq Gr(2,5)$

$$W_q = \frac{}{P_\phi} + \frac{}{P_{\square\square}} + \frac{}{P_{\boxminus}} + \frac{}{P_{\boxplus}} + \frac{}{P_{\boxplus}}$$

denominators $\longleftrightarrow$ cyclically consecutive minors

$\updownarrow$



max. rectangle partitions

$\rightsquigarrow$ numerators: quantum Monk's rule:

quantum mult. in $QH^*(X)$ by $\underset{\sim}{\sigma^{\square}}$ (hyperplane class)

$$\sigma^{\square} * \sigma^{\lambda} = \sum_{\mu = \lambda \cup \square} \sigma^{\mu} + q \sum_{\substack{\lambda_1 = n-k \\ \lambda_k \neq 0}} \sigma^{\lambda_2 - 1, \dots, \lambda_k - 1}$$

Apply to denom:

$$\sigma^{\square} * \sigma^{\phi} = \sigma^{\square} \qquad \sigma^{\square} * \sigma^{(1,1,1)} = \sigma^{(2,1,1)}$$

$$\sigma^{\square} * \sigma^{(3)} = \sigma^{(3,1)} \qquad \sigma^{\square} * \sigma^{(2,2)} = \sigma^{(3,2)}$$

$$\sigma^{\square} * \sigma^{(2,2,2)} = q\,\sigma^{(1,1)}$$

$$\therefore\ W = \frac{P_{\square}}{P_{\phi}} + \frac{P_{(3,1)}}{P_{(3)}} + \frac{P_{(2,1,1)}}{P_{(1,1,1)}} + \frac{P_{(3,2)}}{P_{(2,2)}} + \frac{q P_{(1,1)}}{P_{(2,2,2)}}$$

$X^{\vee} = \{ P_{\phi} P_{(3)} P_{(1,1,1)} P_{(2,2)} P_{(2,2,2)} \neq 0 \}$

- log CY $\therefore$ $\exists$ unique (up to $\exists$ scalar) regular $N = k(n-k)$ form ω on $X^{\vee}$ with log poles along $Gr(n-k, n) \setminus X^{\vee}$

Thm [Marsh-Rietsch] The map $\sigma^{\lambda} \longmapsto [P_{\lambda}\omega]$ defines an injective hom. of v.b. with connection.

Why not isom?: We are not sure whether ~~dim~~ $H^N_{dR}(X^{\vee}, d + dW_q \wedge -)$ has the right dim.

If W_q is "nice", by ~~the~~ a result of Sabbah

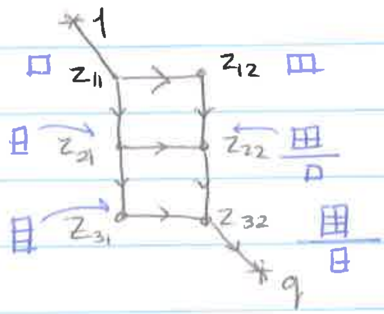
$$\dim H^N_{dR}(-) = \# \operatorname{Crit}(W_q)$$

Remark: W_q always has $\dim H^*(X)$ critical points on $X^{\vee}$

Earlier LG model for $Gr(k,n)$

defined on a torus : $(\mathbb{C}^*)^{k(n-k)}_{z_{ij}}$

$Gr(3,5)$



$$L_q = \sum_{\text{arrow}} \frac{\text{head}}{\text{tail}}$$

$$= z_{11} + \frac{z_{12}}{z_{11}} + \cdots$$

Remark: In general L_q does not have $\dim(H^*(Gr(k,n)))$ Critical pts on $(\mathbb{C}^*)^{k(n-k)}$

e.g. $Gr(2,4)$: L_q has 4 crit points & $\dim H^*(X) = 6$

The 2 missing crit. pts are in $X^\vee$, but in no cluster torus

(2) Quadrics Q_N $N = \dim$

[Givental] constructed an LG model for Q_N

His mirror $\check{X}_{Giv,q} := \left\{ (z_1, \ldots, z_{N+2}) \in (\mathbb{C}^*)^{N+2} \;\middle|\; \begin{array}{l} z_1 \cdots z_{N+2} = q \\ z_{N+1} + z_{N+2} = 1 \end{array} \right\}$

$W_{Giv,q} := z_1 + \cdots + z_{N+2}$

ω_{Giv} = residue from the form on $(\mathbb{C}^*)^{N+2}$

Lie-theoretic mirror : $N = 2m-1$ (odd) : type B $\leftarrow X = Q_N$

type C $\leftarrow X^\vee$

$X = Spin(2m-1)/P$

$X^\vee \in PSp(2m)/P^\vee = \mathbb{P}^{2m-1}$

$H^*(Q_{2m-1}) = \bigoplus_{i=0}^{2m-1} \mathbb{C}\sigma_i \in H^{2i}$

$\leftrightarrow p_0, p_1, \ldots, p_{2m-1}$

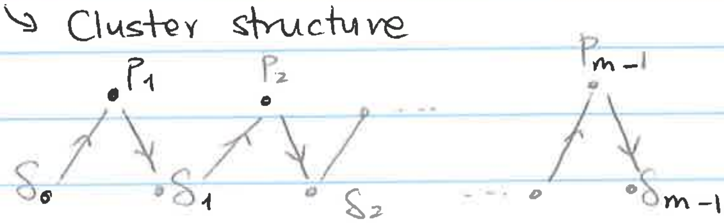
Def: $\quad S_\ell = \sum_{i=0}^{\ell} p_i \, p_{N-\ell-i}$

$$W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-1} \frac{p_{\ell+1}\, p_{N-\ell}}{S_\ell} + q\frac{p_1}{p_{2m-1}}$$

Fact: It recovers $W_{Giv,q}$, but it always has the required number of crit. pts.

$X^\vee = \{\prod \text{denoms}(W) \neq 0\}$

↳ Cluster structure



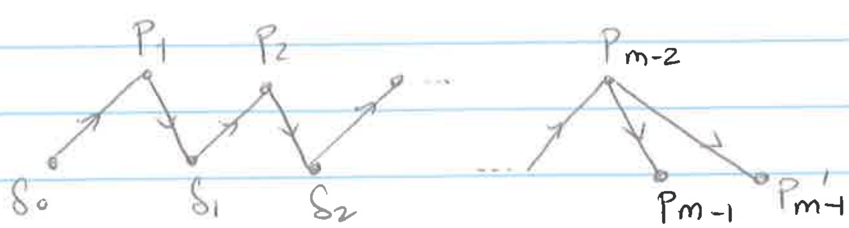
If $N = 2m-2$ even.

$X^\vee \subseteq Q_N^\vee$ 'dual quadric'

$$H^*(X) = \bigoplus_{i=0}^{2m-2} \mathbb{Z}\sigma_i \oplus \mathbb{Z}\sigma'_{m-1}$$

$$Q^\vee_{2m-2} \subseteq \mathbb{P}^{2m-1}_{p_0,\dots,p_{m-1},\, p'_{m-1},\dots,\, p_{2m-1}}$$

$$W_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1}\, p_{N-\ell}}{S_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + q\frac{p_1}{p_{2m-2}}$$



Thm: ω can. N-form $\quad \sigma_i \longmapsto [p_i\omega]$ is inj. form

Rmk: $Q_3 \rightsquigarrow$ iso. [Gorbounov – Smirnov]

(3) $LG(m)$

$(\mathbb{C}^{2m}, \eta)$

symplectic form

$LG(m) = \{ V \subseteq \mathbb{C}^{2m} \mid \dim V = m, \ \eta|_V = 0 \}$

PSp_{2m} $\circlearrowleft$

The mirror lives inside $OG(m) \circlearrowleft Spin(2m+1)$

= max. orth. Grassmn

$H^*(LG)$ = basis 1:1 strict part. $\subseteq$ $m \times m$ box

$\updownarrow$

coord on $OG(m)$

Ex. $LG(3)$

$$W_q = \frac{p_{1}}{p_{21}} + \frac{p_{2}\,p_{3} - p_{32}}{p_{1}\,p_{3} - p_{31}} + \frac{p_{31}\,p_{32} - p_{3}\,p_{321}}{p_{21}\,p_{32} - p_{2}\,p_{321}} + q\,\frac{p_{21}}{p_{321}}$$

Th^m [Pech-Rietsch] It is the Lie-theoretic mirror.