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Stability conditions and cluster varieties

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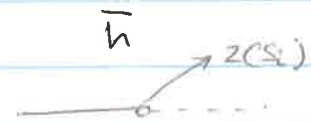
(Q, W) : ~~potenti~~ quiver with nondegenerate potential

$$D_{(Q,W)} = D^b(\text{mod}(\Gamma(Q,W))) \quad \text{3CY category}$$

$$U = \text{mod}(J(Q,W)) \quad S_1, \dots, S_n \text{ form basis for } K(A) \cong K(D) : \Gamma$$

Stability conditions on $D_{(Q,W)}$

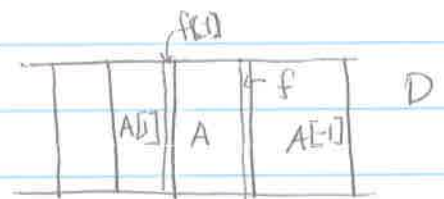
$$\text{Stab}(D_{(Q,W)}) \cong U(A) \cong \bar{h}^\wedge$$



Keller-Yang: There are equivalences, for k -vert. of Q

$$\Phi_k^\pm : D_{(Q,W)} \xrightarrow{\simeq A} D_{M_k(D,W)} \xrightarrow{\simeq A'}$$

$$\Phi_k^\pm(A) = M_k^\pm(A')$$



$$f = \{s_k^{\oplus n} \mid n \in \mathbb{N}\}$$

Say heart \tilde{A} is reachable from a standard heart A if \tilde{A} is obtained by applying finitely many Φ_k^\pm to A .

$$\{\text{reachable hearts}\} \longleftrightarrow \{\text{clusters}\}$$

via twists

$$\text{Stab}(D_{(Q,W)}) \supset \coprod_{\tilde{A} \text{ reachable}} U(\tilde{A}) \cong \bar{h}^\wedge$$

$$X(Q) = \bigcup_{\text{cluster}} (k^*)^n$$

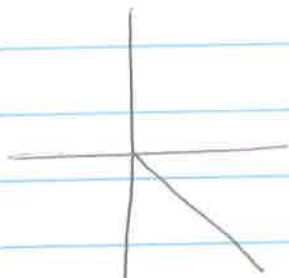
Stability scattering diag for $D_{(Q,W)}$

$(\Gamma \cong K(D) \text{ lattice}, A)$ stand. heart

Walls : $\{ \theta \in \text{Hom}(\Gamma, \mathbb{R}) \mid \exists \theta\text{-stable obj of } A \}$
 \Downarrow
 $\exists Z\text{-stable obj for } Z(x) = -\theta(x)$
 $\arg(Z(x)) = \frac{\pi}{2} + \dim(x) \}$

Functions : $\exp\left(\sum_{d \in \Theta^+} J(d) z^d\right)$

Ex. Walls $\langle [S_i], - \rangle \rightsquigarrow \exp\left(\sum \frac{z^n [S_i] (-1)^i}{n^2}\right)$



Associate to each chamber a heart A_θ , $\theta \in \text{character}$ by tilting A wrt to a torsion pair $(\mathcal{T} = \{ \text{HN-factors wrt } Z_\theta \text{ have phase} \geq 1/2 \})$

Say a reachable heart \tilde{A} is nearby if it can be obtained as a tilt (left) of A wrt some torsion pair

There is canonical choices of signs s.t. any composition of KY equivalences s.t. A is sent to a nearby heart

Define $\mathcal{C}(A) \subset \text{Stab}(\text{Dco}, w)$
 \cup
 $\cup U(\tilde{A})$
 \tilde{A} nearby

Thm (Bridgeland) $\text{Im } Z : \mathcal{C}(A) \rightarrow \text{Hom}(\text{KCD}, \mathbb{R})$
 $\text{Im}(Z)$ is surjective.

$\text{Im}(Z)^{-1}(\text{Walls of stab scattering diag}) \subseteq (\text{Walls in } \mathcal{C}(A) \text{ of second kind})$

$\text{Im}(Z)^{-1}(\text{chamber}) \subseteq U(A_\theta)$

Stability scatt. diag = Cluster scatt. diag
if $\Psi(Q, W)$ is gentle (in part, if Q is acyclic)
 \updownarrow defⁿ

The only objects x of A which are $\langle [x], - \rangle$ -stable
are the simple objects S_i .

Ex. (Q, W) triangulation of marked bordered surface

$\text{Stab}(D_{(Q, W)})$: quadratic differentials

$X(Q)$ moduli space rk 2
local systems.

