Directed homotopy and homology theory with and eye towards applications.

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Directed topology

Topological spaces with *direction*. Many models, here

Definition

Objects of **dTop** are d-Spaces: $(X, \vec{P}(X))$ where $X \in \mathbf{Top}$, $\vec{P}(X) \subseteq X^I$, the dipaths. $\vec{P}(X)$ is

- closed under concatenation,
- contains the constant paths
- closed under reparametrized subpath, i.e., composition with $f : I \to I$ non-decreasing but not necessarily surjective.

A d-map $f : X \to Y$ is a continuous map satisfying
$\gamma \in \vec{P}(X) \Rightarrow f \circ \gamma \in \vec{P}(Y)$

Examples of d-spaces

- $\vec{P}(X)$ the constant paths.
- $\vec{P}(X) = X^I$, all paths.
- $X \subseteq \mathbb{R}^n$ and $\gamma \in \vec{P}(X)$ if $s \leq t \Rightarrow \gamma_i(s) \leq \gamma_i(t)$, $i = 1, \ldots, n$
- $X = \Gamma_1 \times \ldots \times \Gamma_n$ a product of directed graphs. $\gamma \in \vec{P}(X)$ if γ_i is a directed path in Γ_i, $i = 1, \ldots, n$
- Many examples: Directed paths are paths which are locally increasing wrt. a reasonable local order structure.

Relationships between **Top** and **dTop**

- Forgetful functor $U : \mathbf{dTop} \to \mathbf{Top}$.
- Right adjoint: $\vec{P}(X) = X^I$. All continuous maps to $(X, \vec{P}(X))$ are dimaps.
- Left adjoint: $\vec{P}(X) =$ the constant paths. All continuous maps from $(X, \vec{P}(X))$ are dimaps.

Limits and colimits, **dTop** is complete and cocomplete (Grandis)

$(X_i, \vec{P}(X_i))$ d-spaces

- Product $\gamma : I \to X_1 \times X_2$ is a dipath if $\gamma_i \in \vec{P}(X_i)$
- Quotient: $p : X \to Z$ Give Z quotient topology. $\vec{P}(Z) = f(\vec{P}(X))^{cl}$ - closure under concatenation and (directed) reparametrization.

Directed topology - invariants?

$\vec{P}(X)(p,q)$ the space of dipaths from p to q, compact-open topology.

The fundamental category

Components of $\vec{P}(X)(p,q)$ organised in The fundamental *category*

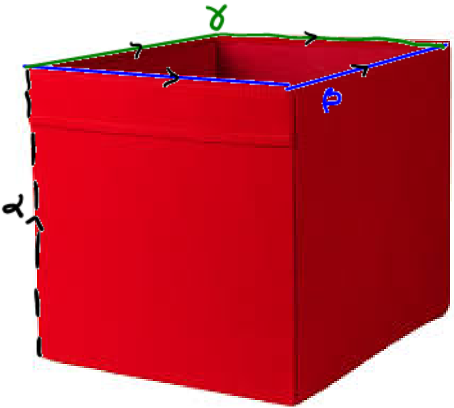
- Objects: All points in X
- Morphisms: $\vec{\pi}_1(X)(p,q)$
  - The connected components of $\vec{P}(X)(p,q)$
  - Equivalently: $\vec{\pi}_1(X)(p,q) = \vec{P}(X)(p,q)/\sim$ where $\sim$ is dihomotopy.

$\vec{\pi}_1(X)(p,q)$ is the number of inequivalent (partial) executions.

OBS: It is *not* a groupoid.

$\vec{\pi}_1 : \mathbf{dTop} \to \mathbf{Cat}$ is a functor - **Cat** is the category of small categories and functors.

No cancellation in $\vec{\pi}_1$



β not dihomotopic to γ

$\beta * \alpha$ is dihomotopic to $\gamma * \alpha$

Directed loops.

In general $\vec{\pi}_1(X)(p,p) \to \pi_1(X)(p,p)$ is neither surjective nor injective.

- a directed loop may be trivial in $\pi_1(X)$.
- There may be non-trivial elements in $\pi_1(X)$ without a directed representative.

Other invariants

(co)Homology, homotopy groups of $\vec{T}(X)(p,q)$, *traces* $\vec{P}(X)(p,q)$ mod reparametrization.

Example

(J.Dubut, E.Goubault, J. Goubault Larrecq, 2016) $\vec{BH}_n(X)$, diagrams with $H_{n-1}(\vec{T}(X)(p,q)$ as vertices. Arrows given by extension maps, pairwise variation of basepoints. $(p,q) \to (p',q')$

Variation of basepoints $\to$ continuous maps $\to$ homomorphisms.
$\sigma \in \vec{P}(X)(p',p)$ induces

$$\sigma^* : \vec{T}(X)(p,q) \to \vec{T}(X)(p',q)$$

.

$$\sigma_* : \vec{T}(X)(r,p') \to \vec{T}(X)(r,p)$$

Other approaches: M.Grandis, 2009, U. Fahrenberg 2013, S. Krishnan, 2013.

Properties of a dihomology theory - wish list

- Invariant under ??
- Will distinguish "enough". In particular, not equivalent to (undirected) homology.
- Does not distinguish "too much". Equivalent dSpaces $\rightarrow$ equivalent dihomology.
- Can be calculated from finite presentations.
- Exact sequences?
- Excision?
- Values in ? (OBS: What are equivalences there?)

Need different theories with different answers.

Equivalences

Dihomeomorphism

$f : X \to Y$ dimap, bijection, inverse $g : Y \to X$ dimap

Dihomotopy equivalence

$f : X \to Y$, $g : Y \to X$, $H : X \times \vec{I} \to X$ dimap, $H(-,0) = g \circ f$, $H(-,1) = Id_X$ OR

- zigzags of such H OR
- I instead of $\vec{I}$,

and similarly for $f \circ g$ and Id_Y.

Bisimulation equivalence

X is bisimilar to Y if there is a d-space Z, $f : Z \to X$, $g : Z \to Y$ (a *span*) s.t. f, g have unique dipath lifting.
Warning: Such maps are called *open* maps in computer science

Inessential equivalence

J.Dubut: $\mathcal{I}(X) \subset \vec{P}(X)$ the maximal Yoneda system - weakly invertibles.

Definition

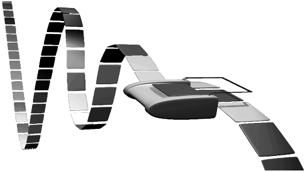
$A \subset X$ is a Future Inessential Deformation Retract if there is a map $H : X \to \mathcal{I}(X)$ s.t. $H(x)(0) = x$, $H(x)(1) \in A$ $H(a)(t) = a$ for $a \in A$ $H_t(x) = H(x)(t)$ is a dimap.

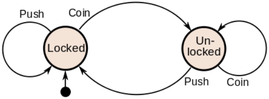
Definition (Inessentially equivalent, J.Dubut)

X and Y are inessentially equivalent if there is a zigzag of Future and Past Inessential Deformation retracts. $X \to Y$

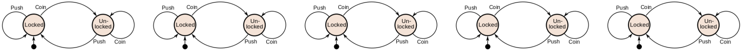
An application/motivation. Concurrent computing

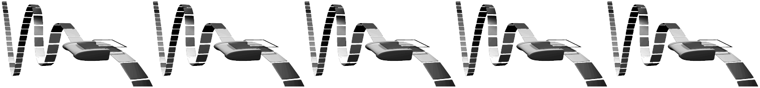
Models of non-parallel programs - threads Turing machine
Graph (automaton).





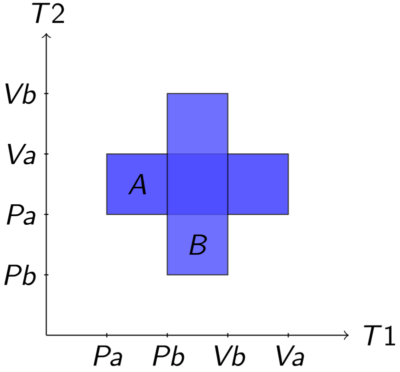
Parallel programs - several models
Several graphs? Several Turing Machines?



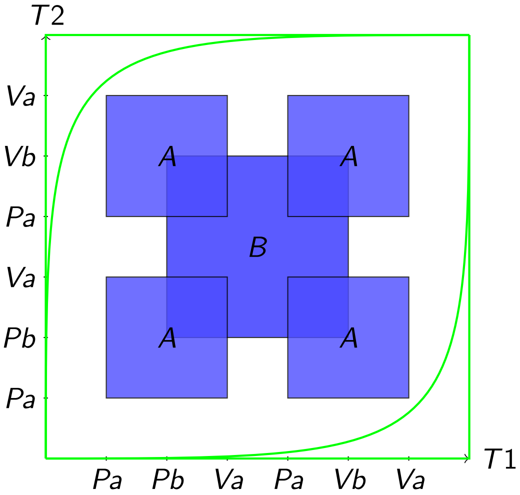


The Swiss flag - two dining philosophers.

$T1 = PaPbVbVa$, $T2 = PbPaVaVb$



An execution is a directed path

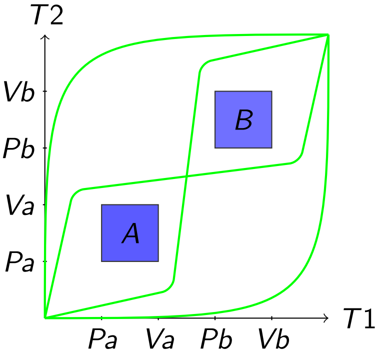


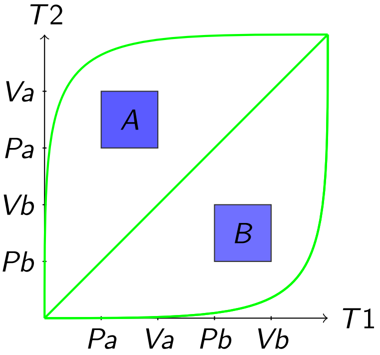
Equivalence of executions

Executions are equivalent, if they have the same output given the same input.

Executions are equivalent, if the corresponding directed paths can be continuously deformed into eachother via directed paths!

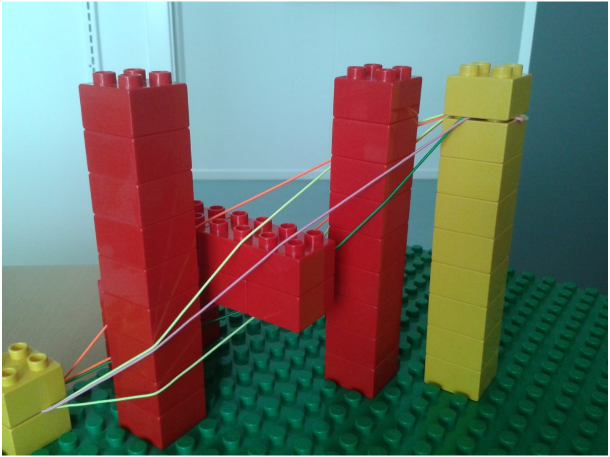
Two holes





The number of dipaths up to equivalence depends on the *order* of the holes.

In 3d, homotopic $\neq$ directed homotopic



Need: Directed topology.

Even in 2d cubical complex, homotopic is not dihomotopic