Applications of monoidal model categories of spectra

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1. A gentle introduction to spectra

2. Applications

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Warning: below "*anything in quotes*" is a heuristic, not a careful definition.

An introduction to stable homotopy theory

"Abelian groups up to homotopy"
spectra $\Longleftrightarrow$ generalized cohomology theories

Examples:

1. **Ordinary cohomology:**

For A any abelian group, $H^n(X;A) = [X_+, K(A,n)]$.

Eilenberg-Mac Lane spectrum, denoted HA.
$HA_n = K(A,n)$ for $n \geq 0$.

The coefficients of the theory are given by

$$HA^*(\mathrm{pt}) = \begin{cases} A & * = 0 \\ 0 & * \neq 0 \end{cases}$$

2. **Hypercohomology:**
For $C.$ any chain complex of abelian groups,
$\mathbb{H}^s(X; C.) \cong \oplus_{q-p=s} H^p(X; H_q(C.))$.
Just a direct sum of shifted ordinary cohomologies.

$HC.^*(\text{pt}) = H_*(C.)$.

3. **Complex K-theory:**
$K^*(X)$; associated spectrum denoted K.

$$K_n = \begin{cases} U & n = \text{odd} \\ BU \times \mathbb{Z} & n = \text{even} \end{cases}$$

$$K^*(\text{pt}) = \begin{cases} 0 & * = \text{odd} \\ \mathbb{Z} & * = \text{even} \end{cases}$$

4. **Stable cohomotopy:**
$\pi_S^*(X)$; associated spectrum denoted $\mathbb{S}$.

$\mathbb{S}_n = S^n$, $\mathbb{S}$ is the *sphere spectrum.*

$\pi_S^*(\text{pt}) = \pi_{-*}^S(\text{pt})$ = stable homotopy groups of spheres. These are only known in a range.

“Rings up to homotopy”
ring spectra $\Longleftrightarrow$ gen. coh. theories with a product

1. For R a ring, HR **is a ring spectrum.**
The cup product gives a graded product:
$HR^p(X) \otimes HR^q(X) \to HR^{p+q}(X)$

Induced by $K(R,p) \wedge K(R,q) \to K(R,p+q)$.

2. For $A.$ a differential graded algebra (DGA), $HA.$ **is a ring spectrum.** Product induced by $\mu : A. \otimes A. \to A.$, or $A_p \otimes A_q \to A_{p+q}$.

The groups $\mathbb{H}(X; A.)$ are still determined by $H_*(A)$, but the product structure is *not* determined $H_*(A)$.

3. K **is a ring spectrum;**
Product induced by tensor product of vector bundles.

4. $\mathbb{S}$ **is a commutative ring spectrum.**

Definition. A "*ring spectrum*" is a sequence of pointed spaces $R = (R_0, R_1, \cdots, R_n, \cdots)$ with compatibly associative and unital products $R_p \wedge R_q \to R_{p+q}$.

Definition. A "*spectrum*" F is a sequence of pointed spaces $(F_0, F_1, \cdots, F_n, \cdots)$ with structure maps $\Sigma F_n \to F_{n+1}$. Equivalently, adjoint maps $F_n \to \Omega F_{n+1}$.

Example: $\mathbb{S}$ a commutative ring spectrum

Structure maps: $\Sigma S^n = S^1 \wedge S^n \xrightarrow{\cong} S^{n+1}$.

Product maps: $S^p \wedge S^q \xrightarrow{\cong} S^{p+q}$.

Actually, must be more careful here. For example: $S^1 \wedge S^1 \xrightarrow{\text{twist}} S^1 \wedge S^1$ is a degree -1 map.

History of spectra and $\wedge$

Boardman in 1965 defined spectra and $\wedge$. $\wedge$ is only commutative and associative up to homotopy.

A_∞ *ring spectrum* = best approximation to associative ring spectrum.

E_∞ *ring spectrum* = best approximation to commutative ring spectrum.

Lewis in 1991: No good $\wedge$ exists.
Five reasonable axioms $\Longrightarrow$ no such $\wedge$.

Since 1997, lots of monoidal categories of spectra exist! (with $\wedge$ that is commutative and associative.)
1. 1997: Elmendorf, Kriz, Mandell, May
2. 2000: Hovey, S., Smith
3, 4 and 5 ... Lydakis, Schwede, ...

Theorem.(Mandell, May, Schwede, S. '01; Schwede '01)
All above models define the same homotopy theory.

Spectral Algebra

Given the good categories of spectra with $\wedge$, one can easily do algebra with spectra.

Definitions:

A *ring spectrum* is a spectrum R with an associative and unital multiplication $\mu : R \wedge R \to R$ (with unit $\mathbb{S} \to R$).

An *R-module spectrum* is a spectrum M with an associative and unital action $\alpha : R \wedge M \to M$.

$\mathbb{S}$-*modules* are spectra.
$S^1 \wedge F_n \to F_{n+1}$ iterated gives $S^p \wedge F_q \to F_{p+q}$.
Fits together to give $\mathbb{S} \wedge F \to F$.

$\mathbb{S}$-*algebras* are ring spectra.

Applications of monoidal model categories of spectra:

We'll focus on applications of Symmetric spectra and Orthogonal spectra.

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Waldhausen's Algebraic K-theory:

Theorem. (Geisser, Hesselholdt '99) Waldhausen's $S_\bullet$-construction for a Waldhausen category C of cofibrations and weak equivalences naturally produces a symmetric spectrum $K(C)$.

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If C is also a symmetric monoidal category with a bi-exact product (i.e. the product interacts well with the cofibrations and weak equivalences), then $K(C)$ is a symmetric ring spectrum.

(Hovey '01) generalized the definition of symmetric spectra to consider spectra over any monoidal model category $\mathcal{C}$. Here instead of a sequence of spaces, one considers a sequence of objects $(C_0, C_1, \cdots, C_n, \cdots)$ in $\mathcal{C}$ with structure maps $K \wedge C_n \to C_{n+1}$.

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Theorem. (Jardine '00) Symmetric spectra of motivic spaces (simplicial presheaves) model the motivic stable category.

Theorem. (Robinson '87, Schwede-S. '03, S. '07, Richter-S. 2017)

(1) $\mathcal{C}h \simeq_{\text{Quillen}} H\mathbb{Z}$-Mod

(2) Assoc. $DGA \simeq_{\text{Quillen}}$ Assoc. $H\mathbb{Z}$-Alg

(3) E_∞ $DGA \simeq_{\text{Quillen}}$ Commutative $H\mathbb{Z}$-Alg

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Central step of the proofs: Each of these statements uses (Hovey '01) to produce the stabilization of the Dold-Kan equivalence between non-negatively graded chain complexes and simplicial abelian groups.

Two applications of $DGA \simeq_{\text{Quillen}} H\mathbb{Z}$-Alg:

1. *Topological equivalences of DGAs.* Consider $\mathbb{S}$-algebra equivalences between $H\mathbb{Z}$-algebra spectra. (See Dugger - S. 2007; Bayindir 2017)

Two applications of $DGA \simeq_{\text{Quillen}} H\mathbb{Z}$-Alg:

1. *Topological equivalences of DGAs.* Consider $\mathbb{S}$-algebra equivalences between $H\mathbb{Z}$-algebra spectra. (See Dugger - S. 2007; Bayindir 2017)

2. The establishment of simple, practical algebraic models for rational G-equivariant stable homotopy theories involves the equivalence of rational DGAs and $H\mathbb{Q}$-algebra spectra. (See Greenlees - S. 2017)

For G a compact Lie group that is not finite, we use orthogonal spectra to model G-equivariant stable homotopy theory.