Homological stability

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MSRI Women in Topology 2017
November 30, 2017

A sequence of spaces and maps $X_1 \to X_2 \to X_3 \to \dots$ satisfies homological stability if $H_i(X_n) \to H_i(X_{n+1})$ is an isomorphism for $n \gg i$. Equivalently, $H_i(X_n) \cong H_i(X_\infty)$ for $n \gg i$, where $X_\infty := \bigcup_n X_n$. (This is because homology commutes with colimits.) This is good because $H_i(X_\infty)$ is often easier to compute.

Theorem 1 (McDuff, Segal, May, ... (1970's)). *If $i < \frac{n}{2}$ then:*

$$H_i(\underbrace{\mathrm{Conf}(n, \mathbb{R}^k)}_{X_n}) \cong H_i(\underbrace{\Omega_0^k S^k}_{X_\infty}).$$

Theorem 2 (Harer (1980), Madsen-Weiss (00's), ...). *If $\mathscr{M}_{g,k}$ denotes the moduli space of genus g Riemann surfaces with k boundary components,*

$$H_i(\mathscr{M}_{g,k}) \cong H_i(\Omega_0^\infty \mathbb{C}\mathrm{P}_{-1}^\infty)$$

for $i \leq \frac{g-2}{3}$.

This is a really powerful theorem that woke up the subject: the LHS is one of the classical objects in mathematics.

Theorem 3 (Szymik-W.). *If BV is Thompson's group $V \cong V_{2,1}$, then $\widetilde{H}_*(BV) = 0$. This the $n = 2$ case of*

$$H_i(BV_{n,k}) \cong H_i(\Omega_0^\infty M_{n-1})$$

where M_{n-1} is the Moore spectrum for $\mathbb{Z}/(n-1)$.

E_2-algebras, stability, and the group completion theorem. The definitions in quotes will be slightly incorrect for expositional purposes.

Definition 4. An "E_2-algebra" is a space X with a homotopy associative, homotopy commutative multiplication $X \times X \to X$. An "E_n-algebra" is more commutative if $n > 2$. An "E_∞-algebra" is as homotopy commutative as possible.

Example 5. Look at $X = \bigsqcup_{n \geq 0} \mathrm{Conf}(n, \mathbb{R}^2)$. Putting two configurations next to each other gives a map

$$\mathrm{Conf}(n, \mathbb{R}^2) \times \mathrm{Conf}(m, \mathbb{R}^3) \to \mathrm{Conf}(n+m, \mathbb{R}^2).$$

So we have a multiplication $X \times X \to X$.

Theorem 6 (Recognition principle). *If X is an E_k-algebra then the group completion GX is $\simeq \Omega^k Y$, for some explicit Y.*

Group completion makes π_0 a group. So if π_0 is not a group, then it has no chance of being a loop space.

Theorem 7 (Group completion theorem, special case)**.** *Suppose $X = \bigsqcup_{n\geq 0} X_n$ is an E_k-algebra for $k \geq 2$, with multiplication $\oplus : X_n \times X_m \to X_{n+m}$. Then*

$$H_*(GX) \cong H_*(\mathbb{Z} \times X_\infty)$$

(same X_∞ as before).

This gives a sequence $X_1 \stackrel{\oplus x}{\to} X_2 \stackrel{\oplus x}{\to} X_3 \to \dots$ for $x \in X_1$.

Theorem 8 (Randal-Williams–W., Kranich)**.** *Given an E_2-algebra X and $x \in X$, there are spaces $W_n(X,x)$ and if $W_n(X,x)$ is $\frac{n-2}{k}$-connected for each n, then*

$$H_i(X) \stackrel{\oplus x}{\to} H_i(X)$$

is an isomorphism for $i \leq \frac{n-1}{k}$.

For example, take $X = \bigsqcup_n X_n$ and $x \in X_1$; then $H_i(X_n) \stackrel{\oplus x}{\to} H_i(X_{n+1})$ is an isomorphism.

Fact: Stability theorems are usually proved like this. (You associate some space and your theorem works if the space is highly connected.)

Summary: Suppose $X = \bigsqcup_{n\geq 0} X_n$ is an E_2-algebra for $k \geq 2$. Then $H_i(X_n) \cong H_i(X_\infty)$ for $n \gg i$ if $W_n(X,x)$ is highly connective. Using the recognition principle and the group completion theorem, $H_i(X_n) \cong H_i(\Omega^k_0 Y)$ for $n \gg i$.

We would like to study further the properties of the spaces $W_n(X,x)$.