Linear correlations of multiplicative functions

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Green–Tao type theorem for multiplicative functions?

Let $\varphi_1, \ldots, \varphi_r \in \mathbb{Z}[X_1, \ldots, X_s]$ linear forms, pairwise non-proportional

$a_1, \ldots, a_r \in \mathbb{Z}$

$h_1, \ldots, h_r : \mathbb{N} \to \mathbb{C}$

and consider the linear correlation

$$\sum_{\mathbf{x} \in \mathbb{Z}^s \cap TK} h_1(\varphi_1(\mathbf{x}) + a_1) \ldots h_r(\varphi_r(\mathbf{x}) + a_r) \qquad (*)$$

where $K \subset [-1, 1]^s$, convex. $(T \to \infty)$

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Question: Can one evaluate (*) in general for multiplicative functions $h_i : \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$); that is, if

$$h_i(nm) = h_i(n)h_i(m) \quad \text{whenever} \quad \gcd(n, m) = 1?$$

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Yes – at least for h_i satisfying the following conditions:

A class of multiplicative functions

Let $\mathcal{M}$ denote the set of multiplicative functions $h : \mathbb{N} \to \mathbb{R}$ such that

1. there is $H \geqslant 1$ s.t. $|h(p^k)| \leqslant H^k$ at all prime powers;
2. $|h(n)| \ll_\varepsilon n^\varepsilon$ for all $n \in \mathbb{N}$, $\varepsilon > 0$;
3. there is $\alpha_h > 0$ such that
$$\frac{1}{x}\sum_{p\leqslant x} |h(p)| \log p \geqslant \alpha_h;$$
4. for all $x' \in (x/(\log x)^C, x)$, C fixed, x sufficiently large, $q \leqslant (\log x)^C$ with $p < \log\log x \Rightarrow p|q$, $(A,q) = 1$, we have:
$$\frac{q}{x}\sum_{\substack{n\leqslant x\\ n\equiv A\ (q)}} h(n) = \frac{q}{x'}\sum_{\substack{n\leqslant x'\\ n\equiv A\ (q)}} h(n) + o\left(\frac{q}{\phi(q)}\frac{1}{\log x}\prod_{\substack{p\leqslant x\\ p\nmid q}}\left(1+\frac{|h(p)|}{p}\right)\right).$$

Lemma (Shiu)

Suppose conditions (1) and (3) hold. If x is sufficiently large (w.r.t. H) , q divisible by all primes $p < \log\log x$ and $\gcd(A, q) = 1$, then

$$\frac{q}{x} \sum_{\substack{n \leqslant x \\ n \equiv A \pmod q}} |h(n)| \ll \frac{q}{\phi(q)} \frac{1}{\log x} \exp\left(\sum_{\substack{p \leqslant x \\ p \nmid q}} \frac{|h(p)|}{p} \right),$$

uniformly in A and q, provided that $q \leqslant x^{1/2}$.

multiplicative functions in arithmetic progressions

Theorem (Granville-Soundararajan). Let $C > 0$ be fixed, $h : \mathbb{N} \to \mathbb{C}$ multiplicative and bounded. For given x consider the primitive characters of conductor $\leqslant (\log x)^C$ and enumerate them as $\chi_1, \chi_2, \dots$ in such a way that

$$\Big|\sum_{n\leqslant x} h(n)\overline{\chi_1}(n)\Big| \geqslant \Big|\sum_{n\leqslant x} h(n)\overline{\chi_2}(n)\Big| \geqslant \dots$$

If x sufficiently large, $x^{1/2} \leqslant X \leqslant x$, $q \leqslant (\log x)^C$ and $k \geqslant 2$, then

$$\frac{q}{X}\sum_{\substack{n\leqslant X\\ n\equiv a(q)}} h(n) = \frac{q}{\phi(q)} \sum_{\substack{\chi\ (q)\\ \text{induced by}\\ \chi_1,\dots,\chi_k}} \chi(a)\frac{1}{X}\sum_{n\leqslant X} h(n)\overline{\chi}(n) + O_{C,\varepsilon}\left(\sqrt{k}(\log x)^{-1+\frac{1}{\sqrt{k}}+\varepsilon}\right).$$

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Note: $h \in \mathcal{M}$ satisfies $\frac{1}{x} \sum_{p \leqslant x} |h(p)| \log p \geqslant \alpha_h$ and, hence,

$$\prod_{\substack{p \leqslant x \\ p \nmid q}} \left(1 + \frac{|h(p)|}{p}\right) \gg_\varepsilon (\log x)^{\alpha_h - \varepsilon},$$

so can take $k = 1 + \lceil \alpha_h^{-2} \rceil$.
We may decompose $h = g * \cdots * g * g'$ into H bounded multiplicative functions (+ one with sparse support) by setting

$$g(p^k) = \begin{cases} h(p)/H & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}.$$

Applying the above with g reduces the stability condition to studying

$$\frac{1}{x} \sum_{n \leqslant x} h(n) \chi(n).$$

for $\chi \pmod q$, $q \leqslant (\log x)^C$.

Write

$$S_f(x) = \frac{1}{x}\sum_{n\leqslant x} f(x).$$

Then the stability-condition is equivalent to:

$$S_{h\chi}(x) \sim S_{h\chi}(x')$$

for $x' \in (x(\log x)^{-C}, x)$ whenever

$$|S_{h\chi}(x)| \asymp S_{|h|}(x)$$

for $\chi \pmod q$ with $1 < q \leqslant (\log x)^C$ and $p < \log\log x \implies p|q$.

Can show:

$$S_{|h|}(x) \sim S_{|h|}(x')$$

for $x' \in (x(\log x)^{-C}, x)$.

Using a Selberg–Delange type argument:

- general divisor functions $d_k = \underbrace{\mathbf{1} * \cdots * \mathbf{1}}_{k}$,
- the function

$$r(n) = \#\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = n\}$$

- the indicator function of the set of sums of two squares
- characteristic function of set of numbers composed of primes that split completely in a given Galois extension $K/\mathbb{Q}$ of finite degree

all belong to $\mathcal{M}$.

For non-negative h: stability holds if the main term in [GS] is determined by trivial character, i.e. if

$$|S_{h\chi}(x)| = o(S_{|h|}(x))$$

for all non-trivial $\chi \pmod q$, $q \leqslant (\log x)^C$.

Example: $h(n) = |\lambda_f(n)|$, λ_f the normalised Hecke-eigenvalue of a primitive holomorphic cusp form of weight $k \in 2\mathbb{N}$ for which

$$|\lambda_f(n)| \leqslant d(n), \quad \sum_{p \leqslant x} |\lambda_f(p)| \log p \geqslant \frac{1}{2} \sum_{p \leqslant x} \lambda_f^2(p) \log p \sim \frac{x}{2}$$

For $\alpha \in [0, 2]$,

$$\#\{p \leqslant x : 0 \leqslant |\lambda_f(p)| \leqslant \alpha\} \log x \sim x\mu(\alpha)$$

[via Halász-Granville-Soundararajan (after decomposing $h\chi$), Sato-Tate, PNT in APs and a consideration of $(t \log p_n)_n \bmod 1$]

Main result

Let $h_1, \ldots, h_r \in \mathcal{M}$, $\varphi_1, \ldots, \varphi_r$ be as before and $w(x) = \log\log x$. Then there are constants $B_1, B_2 > 0$ and $\widetilde{W} : \mathbb{R}_{>0} \to \mathbb{N}$,

$$\forall x : \quad \widetilde{W}(x) = \prod_{p<w(x)} p^{\alpha(p,x)} \leqslant (\log x)^{B_1}, \quad \alpha(p,x) \in \mathbb{N},$$

such that the following asymptotic holds as $T \to \infty$:

$$\frac{1}{\operatorname{vol} TK} \sum_{\mathbf{n} \in \mathbb{Z}^s \cap TK} \prod_{i=1}^{r} h_i(\varphi_i(\mathbf{n}) + a_i) \ =$$

$$\left(\prod_{i=1}^{r} \qquad \frac{1}{T} \sum_{n \leqslant T} h_i(n) \right)$$

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$$\left(\prod_{i=1}^{r} h_i(w_i) \frac{\widetilde{W}}{T} \sum_{\substack{n \leqslant T \\ n \equiv A_i \ (\widetilde{W}(T))}} h_i(n) \right)$$

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$w = \operatorname{lcm}(w_1, \ldots, w_r)$

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$w = \operatorname{lcm}(w_1, \ldots, w_r)$

$$+ \, o\left(\frac{1}{(\log T)^r} \prod_{j=1}^{r} \prod_{p \leqslant T} \left(1 + \frac{|h_i(p)|}{p} \right) \right).$$

$$\frac{1}{\operatorname{vol} TK} \sum_{\mathbf{n} \in \mathbb{Z}^s \cap T\mathfrak{K}} \prod_{i=1}^{r} h_i(\varphi_i(\mathbf{n}) + a_i) = \sum_{\substack{w_1,\ldots,w_r \\ p | w_i \Rightarrow p < w(T) \\ w_i \leqslant (\log T)^{B_2}}} \sum_{\substack{A_1,\ldots,A_r \\ \in (\mathbb{Z}/\widetilde{W}(T)\mathbb{Z})^*}}$$

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$$w=\operatorname{lcm}(w_1,\dots,w_r) \qquad\qquad +\,o\left(\frac{1}{(\log T)^r}\prod_{j=1}^{r}\prod_{p\leqslant T}\left(1+\frac{|h_i(p)|}{p}\right)\right).$$

Write

$$S_f(x)=\frac{1}{x}\sum_{n\leqslant x} f(x).$$

The error term dominates if $|S_{h_i}(x)|=o(S_{|h_i|}(x))$ for some i.

By recent work of Elliott / Tenenbaum, $|S_{h_j}(x)| \asymp S_{|h_j|}(x)$ if there exists $t \in \mathbb{R}$ such that

$$\sum_p \frac{|h_j(p)| - \Re(h_j(p)p^{it})}{p} < \infty,$$

and $|S_{h_j}(x)| = o(S_{|h_j|}(x))$ otherwise.

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Frantzikinakis and Host (2016) studied correlations of bounded complex-valued h_j for which there are t_j and χ_j such that

$$\sum_p \frac{1 - \Re(h_j(p)\chi_j(p)p^{it_j})}{p} < \infty.$$

In the same setting, Klurman (2016) obtains asymptotics for

$$\sum_{n \leqslant x} h_1(P_1(n)) \ldots h_r(P_r(n)).$$

Background

Green and Tao's nilpotent Hardy-Littlewood method

Aim: Asymptotic formulae for linear correlations of arithmetic functions

$\varphi_1, \dots, \varphi_r \in \mathbb{Z}[X_1, \dots, X_s]$ — linear forms, pairwise non-proportional

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$$\sum_{\mathbf{x} \in \mathbb{Z}^s \cap tK} h_1(\varphi_1(\mathbf{x}) + a_1) \dots h_r(\varphi_r(\mathbf{x}) + a_r) = t^s \beta_\infty \prod_p \beta_p + o(t^s),$$

where $K \subset [-1, 1]^s$, convex. (as $t \to \infty$)

Green and Tao's nilpotent Hardy-Littlewood method

Such asymptotic formulae hold, provided

1. there are simultaneous pseudo-random majorants for the functions $h_1|_{\{1,\ldots,T\}},\ldots,h_r|_{\{1,\ldots,T\}}$, $T\to\infty$;
2. as $T\to\infty$,

$$h_i\Big|_{\{1,\ldots,T\}} - \Big(\frac{1}{T}\sum_{m\leqslant T} h_i(m)\Big), \qquad (1\leqslant i\leqslant r),$$

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is orthogonal to nilsequences.
" h_i has no non-trivial large Fourier coefficient."

Family of pseudo-random majorants

$\left\{\nu^{(T)} : \{1, \ldots, T\} \to \mathbb{R}_{>0}\right\}_{T \in \mathbb{N}}$ such that $\nu^{(T)}$ is

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$$\frac{1}{T}\sum_{n\leqslant T}\nu^{(T)}(n) \ll \frac{1}{T}\sum_{n\leqslant T}|h(n)|,$$

3. satisfies the linear forms condition: As $T \to \infty$,

$$\frac{1}{T^d}\sum_{\mathbf{n}\in[1,T]^d}\prod_{i=1}^{t}\nu^{(T)}(\psi_i(\mathbf{n})) = (1+o(1))\prod_{i=1}^{t}\frac{1}{T}\sum_{n\leqslant T}\nu^{(T)}(n)$$

for $\psi_i(\mathbf{n}) = \varphi_i(\mathbf{n}) + a_i$ with $\varphi_i \in \mathbb{Z}[u_1, \ldots, u_s]$ pw linearly independed linear forms where t, d, and the coeff's are bounded.

What are we looking for?

For multiplicative arith. functions one can hope for

$$\nu^{(T)}(m) = \sum_{d \leqslant T^\gamma} 1_{d|m} \lambda_d,$$

where $\gamma \in (0, 1/2)$ can be chosen as small as necessary.

Goldston–Yıldırım, Green–Tao:

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(c) $h(p^{k-1}) \leqslant h(p^k)$ for all prime powers.

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Consider *bad* sets $S(\kappa) = \{m \leqslant T : h(m) > H^\kappa h_\gamma^{(T)}(m)\}$.

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(c) $h(p^{k-1}) \leqslant h(p^k)$ for all prime powers.

(Thus $g = \mu * h$ is non-negative.)

Define $h_\gamma^{(T)} : \{1, \ldots, T\} \to \mathbb{R}_{\geqslant 0}$ via

$$h_\gamma^{(T)}(m) = \sum_{d \leqslant T^\gamma} \mathbf{1}_{d|m}\, g(d).$$

Would like $h(m) \ll h_\gamma^{(T)}(m)$ for $m \leqslant T$.

Consider *bad* sets $S(\kappa) = \{m \leqslant T : h(m) > H^\kappa h_\gamma^{(T)}(m)\}$.

Note that

$$\forall m \exists \kappa : m \in S(\kappa) \backslash S(\kappa+1) \quad \text{thus} \quad h(m) \leqslant \sum_{\kappa \geqslant 0} H^{\kappa+1} 1_{S(\kappa)}(m) h_\gamma^{(T)}(m).$$

With the help of a structure theorem (due to Erdős) for $S(\kappa)$, this yields a pseudo-random majorant

$$|h(n)| \ll \nu^{\sharp}(n)$$

for $h \in \mathcal{M}$ with

$$1 \leqslant |h(p^{k-1})| \leqslant |h(p^k)|\,, \quad (k \geqslant 1).$$

Decomposing h

Suppose $h : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$.

$$h(n) = h^\sharp(n) h^\flat(n),$$

where

$$h^\sharp(p^k) = \max(1, h(p), \ldots, h(p^k)),$$

i.e. $h^\sharp(p^{k-1}) \leqslant h^\sharp(p^k)$, and

$$h^\flat(p^k) = h(p^k)/h^\sharp(p^k) \quad (\leqslant 1)$$

Then

$$h(n) \ll \nu(n) = \nu^\sharp(n)\nu^\flat(n)$$

if

$$h^\flat(n) \ll \nu^\flat(n).$$

Majorants for bounded multiplicative functions

Note: If $n = m_1 m_2$ and $(m_1, m_2) = 1$, then $h(n) \leqslant h(m_1)$.

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$$n = p_1^{a_1} \dots p_k^{a_k}, \quad \prod_{i<j} p_i^{a_i} > T^\gamma$$

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$$\nu^\flat(n) = \sum_{Q \leqslant T^\gamma} \sum_{\substack{m \leqslant T^\gamma \\ p|m \Rightarrow p<Q}} \mathbf{1}_{Qm|n} \, \mathbf{1}_{Qm>T^\gamma} \, \mathbf{1}_{(p|\frac{n}{Qm} \Rightarrow p>Q)} \, h(Qm) \, .$$

Green and Tao's nilpotent Hardy-Littlewood method

1. there are simultaneous pseudo-random majorants for the functions $h_1|_{\{1,\dots,t\}}, \dots, h_r|_{\{1,\dots,t\}}$, $t \to \infty$;
2. as $t \to \infty$,

$$h_i\Big|_{\{1,\dots,t\}} - \Big(\frac{1}{t}\sum_{m \leqslant t} h_i(m)\Big), \qquad (1 \leqslant i \leqslant r),$$

   is orthogonal to nilsequences.
   " h_i has no non-trivial large Fourier coefficient."

Combined with a proof of part 2, this allows us to evaluate

$$\sum_{\mathbf{x} \in \mathbb{Z}^s \cap TK} h_1(\varphi_1(\mathbf{x}) + a_1) \dots h_r(\varphi_r(\mathbf{x}) + a_r)$$

with $h_1, \dots, h_r \in \mathcal{M}$.