MOMENTS OF QUADRATIC DIRICHLET L-FUNCTIONS IN FUNCTION FIELDS

1. Number field Case

$$\sum_{|d|\leq X} L(\frac{1}{2}, \chi_d)^k \sim a_k g_k x(\log x)^{\frac{k(k+1)}{2}}, \tag{1}$$

where the sum is over real primitive characters χ_d character mod d, a_k is an Euler product and g_k is a constant coming from Random matrix theory.

Known results:

(1) k=1 Jutila
(2) k=2 Jutila and Soundararajan
(3) k=3 Soundararajan
(4) k=4 possibly on GRH, using ideas of Soundararajan and Young

FKRS

$$M_k = XP_k(\log x) + o(X), \tag{2}$$

where $P_k(x)$ has degree $k(k+1)/2$.

2. Function field case

2.1. background in function field.

q a prime $\equiv 1 \pmod 4$

$$\begin{aligned} \mathbb{F}_q[x] &\leftrightarrow \mathbb{Z} \\ \text{monic irreducible polynomials} &\leftrightarrow \text{ primes} \\ |f| = q^{\deg f} &\leftrightarrow |n| \\ \sum_{f \text{ monic } \deg f \leq n} a(f) &\leftrightarrow \sum_{n\leq X} a(n) \end{aligned}$$

Notation :
$\mathcal{M}_n :=$ monic polynomials of degree n, $|\mathcal{M}_n| = q^n$.
$\mathcal{H}_n :=$ monic, square free polynomials of degree n, $|\mathcal{H}_n| = q^n(1-\frac{1}{q})$

Zeta function:

$$\zeta_q(s) = \sum_{f \text{monic}} \frac{1}{|f|^s} = \sum_{n=0}^{\infty} \frac{q^n}{q^{ns}} = \frac{1}{q^{1-s}}. \text{ poles } s=1, \text{ no zeros}$$
$$Z(u) = \sum_{f \text{ monic}} u^{\deg f} = \frac{1}{1-qu}, \text{ poles at } u = \frac{1}{q}$$
$$\chi_D(f) = \left(\frac{D}{f}\right), \text{ Legendre symbol}$$
$$L(u, \chi_D) := \sum_{f \text{ monic}} \chi_D(f) u^{\deg f}$$

If $D \neq \square$, $L(u, \chi_D)$ is a polynomial.

- If $D \in \mathcal{H}_{2g+1}$, $L(u, \chi_D)$ is a polynomials of deg $2g$.
- Functional equation: $L(u, \chi_D) = (qu^2)^g L(\frac{1}{qu}, \chi_D)$
- RH: All zeros lie on $|u| = \frac{1}{\sqrt{q}}$

2.2. **Function Field moment problem.** Find asymptotic formulas for

$$M_k = \sum_{D \in \mathcal{H}_{2g+1}} L(\frac{1}{2}, \chi_D)^k,$$

as $q^{2g+1} \to \infty$

- Fix g, let $q \to \infty$, Katz and Sarnak
- Fix q, let $g \to \infty$,
  - $k = 1$: Andrade-Keating

    $$M_1 = q^{2g+1} P_1(2g+1) + O(q^{\frac{3g}{2}(1+\epsilon)})$$

    Hoffstein, Rosen

    $$M_1 = q^{2g+1} P_1(2g+1) + O(q^{g(1+\epsilon)})$$

    Florea

    $$M_1 = q^{2g+1} P_1(2g+1) + q^{\frac{2g+1}{3}} Q_1(2g+1) + O(q^{\frac{g}{2}(1+\epsilon)})$$
  - $k = 2, 3$

    $$M_k = q^{2g+1} P_k(2g+1) + O(q^{\frac{kg}{2}(1+\epsilon)}), \deg P_k = \frac{k(k+1)}{2}$$
  - $k = 4$

    $$M_4 = q^{2g+1}(a_{10} g^{10} + a_9 q^9 + a_8 q^8) + O(q^{2g+1} g^{7+\frac{1}{2}+\epsilon})$$

conjectures:(Andrade-Keating Recipe)

$$M_k = q^{2g+1} P_k(2g+1) + o(q^{2g+1}), \deg P_k = \frac{k(k+1)}{2}$$

Recipe

$$\sum_{D \in \mathcal{H}_{2g+1}} L(\frac{1}{2}+\alpha_1, \chi_D) \cdots L(\frac{1}{2}+\alpha_k, \chi_D)$$

$$\Lambda(s, \chi_D) = \Lambda(1-s, \chi_D)$$

$$\sum_{D \in \mathcal{H}_{2g+1}} \Lambda(\frac{1}{2}+\alpha_1, \chi_D) \cdots \Lambda(\frac{1}{2}+\alpha_k, \chi_D)$$

Properties:

– Symmetric under permutation of α_i

– $\alpha_i \to -\alpha_i$

$$\sum_{D \in \mathcal{H}_{2g+1}} \sum_{f_1 \dots f_k} \frac{\chi_D(f_1, \dots, f_k)}{\prod_i |f_i|^{\frac{1}{2}+\alpha_i}}$$

$f_1 \cdots f_k = l^2$,

$$\frac{1}{|\mathcal{H}_{2g+1}|} \to \prod_{p|l} (1+\frac{1}{|p|})^{-1} =: a(l)$$

$$\begin{aligned}
&\sum_{D \in \mathcal{H}_{2g+1}} \sum_{f_1 \dots f_k} \frac{\chi_D(f_1, \dots, f_k)}{\prod_i |f_i|^{\frac{1}{2}+\alpha_i}} \\
&= \sum_{l \text{monic}} a(l) \sum_{f_1 \cdots f_k = l^2} \frac{1}{\prod_i |f_i|^{\frac{1}{2}+\alpha_i}} \\
&= \prod_p \left(1 + \frac{1}{1+\frac{1}{|p|}} \sum_{1 \le i \le j \le k} \frac{1}{|p|^{1+\alpha_i+\alpha_j}} + \cdots \right) \\
&= \prod_{1 \le i \le j \le k} \zeta(1+\alpha_i+\alpha_j) A(\alpha_1, \dots, \alpha_k)
\end{aligned}$$

2.3. **Proof of moments results.**

- Approximate functional equation

$$\sum_{D \in \mathcal{H}_{2g+1}} L(\frac{1}{2}, \chi_D)^k = 2 \sum_{\deg f \le kg} \sum_{D \in \mathcal{H}_{2g+1}} \frac{\chi_D(f) d_k(f)}{\sqrt{|f|}}$$

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$$\sum_{D \in \mathcal{H}_{2g+1}} \chi_D(f) = \sum_{C|p^\infty} \sum_{h \in \mathcal{M}_{2g+1-2\deg C}} \chi_f(h) - q \sum_{C|p^\infty} \sum_{h \in \mathcal{M}_{2g-1-2\deg C}} \chi_f(h)$$

$$\sum \chi_D(f)u^{\deg D} = \prod_{p\nmid f}(1+\chi_p(f)u^{\deg p}) = \frac{L(u,\chi_f)}{L(u^2,\chi_f^2)} = \frac{(1-qu^2)L(u,\chi_f)}{\prod_{p|f}(1-u^{2\deg p})} = \sum_{C|p^\infty} u^{2\deg C}$$

Use Poisson summation formula to evaluate

$$\sum_{\deg f\leq kg} \frac{d_k(f)}{\sqrt{|f|}} \sum_{C|p^\infty} \sum_{h\in\mathcal{M}_{2g+1-2\deg C}} \chi_f(h)$$

Poisson summation formula:

- $a\in\mathbb{F}_q((\frac{1}{x})), a=\sum a_i(\frac{1}{x})^i$, $e(a)=e^{\frac{2\pi i tr(a_1)}{q}}$.
- $G(V,\chi_f)=\sum_{u \pmod f}\chi_f(u)e(\frac{uV}{f})$

$\deg f = n$ even,

$$\sum_{h\in\mathcal{M}_m}\chi_f(h) = \frac{q^m}{|f|}\left[G(0,\chi_f)+(q-1)\sum_{\deg(V)\leq n-m-2}G(V,\chi_f) - \sum_{\deg(V)=n-m-1}G(V,\chi_f)\right]$$

$$q^{2g+1}\sum_{\deg f\leq kg, \deg f even}\frac{d_k(f)}{|f|}\sum_{C|f^\infty}\frac{1}{|C|^2}\left[\frac{G(0,\chi_f)}{\sqrt{|f|}} + \sum_{\deg(V)\simeq \deg f-2g+2\deg C}\frac{G(V,\chi_f)}{\sqrt{|f|}}\right]$$

- $V=0$, $G(0,\chi_f)\neq 0 \iff f=\square$. $f=l^2$,

$$q^{2g+1}\sum_{\deg f\leq kg/2}\frac{d_k(f^2)}{|f|}\prod_{p|l}(1+\frac{1}{|p|})^{-1} = q^{2g+1}P_1(2g+1)+Error$$

$$\sum_l d_k(f^2)u^{\deg f} = \prod_q(1+\frac{k(k+1)}{2}u^{\deg p}+\dots) = Z(u)^{\frac{k(k+1)}{2}}G_k(u)$$

$k=1,\ q^{2g+1}P_1(2g+1)+O(q^g)$

$k=3,4,\ q^{2g+1}P_2(2g+1)$

$k=2,\ q^{2g+1}g$

$k=1,\ q^g$

- $V\neq 0$, Heuristically,

$$\begin{aligned}
&\deg(f) - 2g + 2\deg(C) > 0\\
&\deg(C) < \deg(f)\\
\implies &\deg(f) > \frac{2g}{3}\\
&\frac{G(l^2,\chi_f)}{\sqrt{|f|}} \sim 1\\
\to &\frac{2g}{3} \leq \deg f \leq g\\
\to &q^{\frac{2g+1}{3}} Q(2g+1)
\end{aligned}$$

$$\sum_f \frac{G(V,\chi_f) d_k(f)}{\sqrt{|f|}} w^{\deg f} \sim L(w,\chi_f)^k$$

$$q^{2g+1}\frac{1}{2\pi i}\oint_{|w|=\frac{1}{\sqrt{|f|}}} \sum_{\deg f\leq kg} \frac{1}{w^{\deg f}|f|} \sum_{V\in\mathcal{H}_{2g+1}} L(w,\chi_D)^k \prod_{p|V} \mathcal{A}_p(w)\frac{dw}{w}$$

– $k = 1,2,3 \ll q^{\frac{kq}{2}(1+\epsilon)}$
– $k = 4$, upper bounds

$$\sum_{D\in\mathcal{H}_{2g+1}} |L(w,\chi_D)|^4 \ll q^{2g+1} g^{4+\epsilon} \left(\min\{g, \frac{1}{Q}\}\right)^6,$$

This leads to a bound of size $q^{2g+1}g^{9+\epsilon}$. Write $w = \frac{1}{\sqrt{q}}e^{i\theta}$, then for $|\theta| \leq \frac{1}{q}$, use upper boudn $q^{2g+1}g^{10+\epsilon,}$. For $\theta \sim \alpha$, use upper boudn $q^{2g+1}g^{4+\epsilon}\alpha^{-6}$. Take $\alpha \sim \log\log g$,

$$\begin{aligned}
\sum_{\deg\leq 4g} \frac{d_4(f)\chi_D}{\sqrt{|f|}} &= \sum_{\deg\leq 4g-\alpha} \frac{d_4(f)\chi_D}{\sqrt{|f|}} + \sum_{4g-\alpha<\deg f\leq kg} \frac{d_4(f)\chi_D}{\sqrt{|f|}}\\
&= q^{2g+1}P_{10}(4g-\alpha) + \frac{1}{2\pi i}\oint_{|u|=1} \frac{1-u^\alpha}{(1-u)u^{4g}} \sum_{D\in\mathcal{H}_{2g+1}} L(\frac{u}{\sqrt{|q|}},\chi_D)\frac{du}{u} + O(g^{9-\frac{1}{2}+\epsilon})\\
&= q^{2g+1}(a_{10}1^{10} + b_9 g^9\alpha + a_9 q^9 + b_8 q^8\alpha^2 + \dots - b_9 g^9 \alpha - b_8 q^8 \alpha^2 + \dots\\
&= q^{2g+1}(a_{10}1^{10} + g^9 + \dots)
\end{aligned}$$

- Can get down to g^4 by iterating the process.
- $V \neq \Box$ term doesn't contribute to 4-the moment, but will contribute for $k > 4$.