Determination of elliptic curves by their adjoint p-adic L-functions

Maria Nastasescu

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Based on:

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Complex L-function of an elliptic curve

- Given $E/\mathbb{Q}$ an elliptic curve with conductor N, define the complex L-function of E by the Euler product for $\mathrm{Re}(s) > \frac{3}{2}$:

$$L(E,s) = \prod_{r|N} \frac{1}{1 - a_r r^{-s}} \prod_{r\nmid N} \frac{1}{1 - a_r r^{-s} + r^{1-2s}}$$

where $a_r = r + 1 - \#E(\mathbb{F}_r)$ if $r \nmid N$.

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  where $a_r = r + 1 - \#E(\mathbb{F}_r)$ if $r \nmid N$.

- If $r|N$ then a_r depends on the reduction of E at r. More specifically, $a_r = 1$ if E has split multiplicative reduction at r, $a_r = -1$ if E has non-split multiplicative reduction at r and $a_r = 0$ if E has additive reduction at r.

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- $L(E,s)$ extends to a holomorphic function on $\mathbb{C}$ satisfying a functional equation between s and $2-s$.

Complex adjoint L-function of an elliptic curve

- The symmetric square L-function of E for $\mathrm{Re}(s) > 2$:

$$L(Sym^2 E, s) = \prod_{r \text{ prime}} P_r(r^{-s})^{-1}.$$

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$$P_r(X) = (1 - \alpha_r^2 X)(1 - \beta_r^2 X)(1 - rX).$$

Here α_r and β_r are the roots of the polynomial

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  Here α_r and β_r are the roots of the polynomial

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  with a_r the trace of the Frobenius at r.
- Let χ be an even Dirichlet character. Then $L(Sym^2 E, \chi, s)$ has a holomorphic continuation over $\mathbb{C}$ and satisfies a functional equation between s and $3 - s$.

Determining elliptic curves by their L-functions

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- **Question:** Does the symmetric square L-function of E determine the elliptic curve? What about its p-adic analogue?
- **Answer:** The values $\{L_p(Sym^2 E, c_n\}_{n \geq 0}$ determine E up to quadratic twists.

Main result

Theorem (N.)

Let E, E' be elliptic curves over $\mathbb{Q}$ with semistable reduction at p. Suppose

$$L_p(Sym^2 E, n) = CL_p(Sym^2 E', n)$$

for infinitely many integers n prime to p and some constant $C \in \overline{\mathbb{Q}}$. Then E' is isogenous to a quadratic twist E_D of E. If E and E' have square free conductors, then E and E' are isogenous over $\mathbb{Q}$.

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- We set

$$L_u(\pi, s) = (2\pi)^{-s-1/2} L\left(E, s + \frac{1}{2}\right).$$

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- $Sym^2(\pi) \cong I_K^{\mathbb{Q}}(\eta^2) \boxplus \eta_0$, with η_0 the restriction of η to $\mathbb{Q}$.
- $L(Sym^2\pi, s) = L(\pi', s)L(\eta_0, s)$, with $\pi' = I_K^{\mathbb{Q}}(\eta^2)$ cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$

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- **Step 3:** $L(\pi \otimes \chi, 1, sym^2) \neq 0$ for infinitely many χ p-power characters.
- **Step 4:** The values $\{L(\pi \otimes \chi, 1, sym^2)\}$ with χ a p-power character determine π up to twists by quadratic characters.

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- **Dabrowski-Delbourgo:** construct a p-adic analogue to $L(Sym^2 E, \chi, s)$ with $s \in \mathbb{Z}_p$ by the Mellin transform of a p-adic distribution μ_p on $\mathbb{Z}_p^\times$.

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- This distribution μ_p is defined by interpolating the values of the complex symmetric square L-function at all twists by Dirichlet characters of p-power order

Višik's theory of *h*-admissible measures

- Assume that μ takes values in $\mathbb{C}_p$. We say that μ is a **bounded measure** if

$$\left| \int_{a+p^n\mathbb{Z}_p} \mathrm{d}\mu \right|_p$$

is bounded for all $n \in \mathbb{N}$ and $(a, p) = 1$.

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- The set of *h*-admissible measures with $h = 1$ is strictly larger, but contains the bounded measures.

Adjoint p-adic L-function of an elliptic curve

Suppose first that E has good reduction at p.

- We define a distribution μ_p on E such that

$$\int_{\mathbb{Z}_p^\times} \chi \mathrm{d}\mu_p = C_E \cdot \alpha_p(E)^{-2m_\chi} \tau(\overline{\chi})^2 p^{m_\chi} L(Sym^2 E, \chi, 2)$$

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- If E has good ordinary reduction at p, the distribution μ_p is a bounded measure on $\mathbb{Z}_p^\times$.
- If E has supersingular reduction at p then μ_p is a 2-admissible measure.

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Suppose now that E has bad multiplicative reduction at p.

- We define a distribution μ_p on E such that

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Define

$$L_p(Sym^2 E, \chi, s) := \int_{\mathbb{Z}_p^\times} \chi(x) \langle x \rangle^s \mathrm{d}\mu_p$$

where $\langle \cdot \rangle : \mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p$, $\langle x \rangle = \frac{x}{\omega(x)}$ with $\omega : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ the Teichmüller character.

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Lemma

If $L_p(Sym^2 E, n) = CL_p(Sym^2 E', n)$ *for infinitely many* $(n, p) = 1$ *then*

$$L(Sym^2 E, \chi, 2) = C_1 C_2^a L_p(Sym^2 E', \chi, 2)$$

for infinitely many χ *p-power characters of conductor* p^a.

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- Can reduce the proof of the main result to showing that if π is a cuspidal automorphic representation of $GL(3, \mathbb{A}_{\mathbb{Q}})$

  $$L(\pi \otimes \chi, 1) \neq 0$$

  for infinitely many Dirichlet *p*-power characters, and that moreover these values determine π up to isomorphism.

Motivation

F a number field, π a cuspidal automorphic representation of $\mathrm{GL}(n, \mathbb{A}_F)$.

- **Rohrlich**: For $n = 2$, there exist infinitely many ray class characters χ of F such that $L(\pi \otimes \chi, s_0) \neq 0$, for any fixed $s_0 \in \mathbb{C}$.

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- Later work focused on twists by sparser sets of characters.

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Theorem (N.)

Suppose π, π' are two unitary cuspidal representations of $GL(3, \mathbb{A}_{\mathbb{Q}})$ with the same central character ω. Suppose there exist $B, C \in \mathbb{C}$ such that

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for some fixed $1 \geq \beta > \frac{2}{3}$ and for all p-power order characters of conductor p^a for all but finitely many a. Then $\pi \cong \pi'$.

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Note: if π, π' are tempered unitary cuspidal automorphic representations then the same result holds if (1) is satisfied for some fixed $1 \geq \beta > \frac{1}{2}$. (If the generalized Ramanujan conjecture is true then this condition is automatically satisfied)

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Suppose π, π' are two unitary cuspidal automorphic representations of $GL(2, \mathbb{A}_{\mathbb{Q}})$ with the same central character ω. Suppose there exist constants $B, C \in \mathbb{C}$ such that

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- We show that $\{\alpha_l \, \beta_l, \gamma_l\} = \{\alpha'_l, \beta'_l, \gamma'_l\}$ and then apply the Strong Multiplicity One Theorem to obtain the result.

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- A p-adic L-function was constructed for certain automorphic representations π of $\mathrm{GL}(2n, \mathbb{A}_{\mathbb{Q}})$ under certain conditions.

Thank you!