Systems of quadratic and cubic diagonal equations

Julia Brandes

Chalmers Institute of Technology / University of Gothenburg

February 03, 2017

The question

Consider systems of diagonal equation

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \le i \le r).$$

What is the number $N(P)$ of integral solutions $1 \le x_1, \ldots, x_s \le P$ to such a system?

The question

Consider systems of diagonal equation

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \leq i \leq r).$$

What is the number $N(P)$ of integral solutions $1 \leq x_1, \ldots, x_s \leq P$ to such a system?

Heuristics:

- Typically expect $N(P) \approx P^{s - \sum k_i}$.
  - if equations are suff. independent $\rightarrow$ non-singularity conditions!
  - possible obstructions to real or p-adic solubility

The question

Consider systems of diagonal equation

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \le i \le r).$$

What is the number $N(P)$ of integral solutions $1 \le x_1, \ldots, x_s \le P$ to such a system?

Heuristics:

- Typically expect $N(P) \approx P^{s-\sum k_i}$.
  - if equations are suff. independent $\to$ non-singularity conditions!
  - possible obstructions to real or p-adic solubility
- "square root barrier": without further info on coefficients, can get $N(P) \gg P^{s/2}$.

The question

Consider systems of diagonal equation

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \leq i \leq r).$$

What is the number $N(P)$ of integral solutions $1 \leq x_1, \ldots, x_s \leq P$ to such a system?

Heuristics:

- Typically expect $N(P) \approx P^{s-\sum k_i}$.
  - if equations are suff. independent $\rightarrow$ non-singularity conditions!
  - possible obstructions to real or p-adic solubility
- "square root barrier": without further info on coefficients, can get $N(P) \gg P^{s/2}$.

$\rightarrow$ expect $N(P) \sim cP^{s-\sum k_i}$ for $s \geq 2\sum k_i + 1$.

What do we know?

Question

For what systems

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \le i \le r)$$

can we show that

$$N(P) \sim cP^{s - \sum k_i} \quad \text{for} \quad s \ge 2\sum k_i + 1?$$

What do we know?

Question

For what systems

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \le i \le r)$$

can we show that

$$N(P) \sim cP^{s-\sum k_i} \quad \text{for} \quad s \ge 2\sum k_i + 1?$$

- Systems of linear and quadratic forms: classical.

What do we know?

Question

For what systems

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \le i \le r)$$

can we show that

$$N(P) \sim cP^{s-\sum k_i} \quad \text{for} \quad s \ge 2\sum k_i + 1?$$

- Systems of linear and quadratic forms: classical.
- Vinogradov systems: $k_i = i$ for $1 \le i \le K$. (follows from VMVT – see BDG)

What do we know?

Question

For what systems

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \leq i \leq r)$$

can we show that

$$N(P) \sim cP^{s - \sum k_i} \quad \text{for} \quad s \geq 2\sum k_i + 1?$$

- Systems of linear and quadratic forms: classical.
- Vinogradov systems: $k_i = i$ for $1 \leq i \leq K$. (follows from VMVT – see BDG)
- One quadratic and one cubic equation – Wooley 2015 (uses cubic VMVT)

What do we know?

Question

For what systems

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \le i \le r)$$

can we show that

$$N(P) \sim cP^{s-\sum k_i} \quad \text{for} \quad s \ge 2\sum k_i + 1?$$

- Systems of linear and quadratic forms: classical.
- Vinogradov systems: $k_i = i$ for $1 \le i \le K$. (follows from VMVT – see BDG)
- One quadratic and one cubic equation – Wooley 2015 (uses cubic VMVT)
- Systems of r cubic equations: $N(P) \gg cP^{s-3r}$ whenever $s \ge 6r+1$ (Brüdern–Wooley 2016).

What do we know?

Question

For what systems

$$c_{i,1}x_1^{k_i} + \cdots + c_{i,s}x_s^{k_i} = 0 \quad (1 \le i \le r)$$

can we show that

$$N(P) \sim cP^{s-\sum k_i} \quad \text{for} \quad s \ge 2\sum k_i + 1?$$

- Systems of linear and quadratic forms: classical.
- Vinogradov systems: $k_i = i$ for $1 \le i \le K$. (follows from VMVT – see BDG)
- One quadratic and one cubic equation – Wooley 2015 (uses cubic VMVT)
- Systems of r cubic equations: $N(P) \gg cP^{s-3r}$ whenever $s \ge 6r + 1$ (Brüdern–Wooley 2016).

This talk:

- One cubic and r_2 quadratic equations,
- One quadratic and r_3 cubic equations.

Warm-up: One quadratic and one cubic equation

Wooley's theorem ($r_2 = r_3 = 1$) by the circle method: Count solutionns $1 \leq x_i \leq P$ to

$$c_1 x_1^2 + \cdots + c_s x_s^2 = 0$$
$$d_1 x_1^3 + \cdots + d_s x_s^3 = 0.$$

Warm-up: One quadratic and one cubic equation

Wooley's theorem ($r_2 = r_3 = 1$) by the circle method: Count solutionns $1 \leq x_i \leq P$ to

$$\begin{aligned} c_1 x_1^2 + \cdots + c_s x_s^2 &= 0 \\ d_1 x_1^3 + \cdots + d_s x_s^3 &= 0. \end{aligned}$$

Let

$$f(\alpha_2, \alpha_3) = \sum_{x=1}^{P} e(\alpha_2 x^2 + \alpha_3 x^3).$$

Then by orthogonality,

$$N(P) = \sum_{x_1, \dots x_s} \int_0^1 e(\alpha \sum c_i x_i^2) \mathrm{d}\alpha \int_0^1 e(\beta \sum d_i x_i^3) \mathrm{d}\beta$$

Warm-up: One quadratic and one cubic equation

Wooley's theorem ($r_2 = r_3 = 1$) by the circle method: Count solutionns $1 \leq x_i \leq P$ to

$$\begin{aligned} c_1 x_1^2 + \cdots + c_s x_s^2 &= 0 \\ d_1 x_1^3 + \cdots + d_s x_s^3 &= 0. \end{aligned}$$

Let

$$f(\alpha_2, \alpha_3) = \sum_{x=1}^{P} e(\alpha_2 x^2 + \alpha_3 x^3).$$

Then by orthogonality,

$$N(P) = \sum_{x_1, \ldots x_s} \int_0^1 e(\alpha \sum c_i x_i^2) \mathrm{d}\alpha \int_0^1 e(\beta \sum d_i x_i^3) \mathrm{d}\beta = \int_{[0,1]^2} \prod_{i=1}^{s} f(c_i \alpha, d_i \beta) \mathrm{d}\alpha \mathrm{d}\beta.$$

Need to understand

$$N(P) = \int_{[0,1]^2} \prod_{i=1}^{s} f(c_i\alpha, d_i\beta)\mathrm{d}\alpha\mathrm{d}\beta.$$

Need to understand

$$N(P) = \int_{[0,1]^2} \prod_{i=1}^{s} f(c_i\alpha, d_i\beta)\mathrm{d}\alpha\mathrm{d}\beta.$$

Idea:

dissect unit interval in

- major arcs $\mathfrak{M}$, where the integrand is large $\rightarrow$ main term,
- minor arcs $\mathfrak{m}$, where the integrand is small $\rightarrow$ error term.

Need to understand

$$N(P) = \int_{[0,1]^2} \prod_{i=1}^{s} f(c_i\alpha, d_i\beta)\mathrm{d}\alpha\mathrm{d}\beta.$$

Idea:

dissect unit interval in

- major arcs $\mathfrak{M}$, where the integrand is large $\rightarrow$ main term,
- minor arcs $\mathfrak{m}$, where the integrand is small $\rightarrow$ error term.

Take $[0,1]^2 = \mathfrak{M} \cup \mathfrak{m}$, where

$$\mathfrak{M} = \{\boldsymbol{\alpha} \in [0,1]^2 : |q\alpha_i - a_i| \leq P^{-9/4}, q \leq P^{3/4}, \quad a_i, q_i \in \mathbb{N}\}.$$

By fairly standard methods ($s \geq 11$):

$$N(P; \mathfrak{M}) = \int_{\mathfrak{M}} \prod_{i=1}^{s} f(c_i\alpha_2, d_i\alpha_3)\mathrm{d}\boldsymbol{\alpha} \sim cP^{s-5}.$$

Need to understand contribution from minor arcs.
Let $s \geq 11$, then

$$N(P;\mathfrak{m}) = \int_{\mathfrak{m}} \prod_{i=1}^{s} f(c_i\alpha_2, d_i\alpha_3)\mathrm{d}\boldsymbol{\alpha} \ll \sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})|^{s-10} \int_{[0,1]^2} |f(\boldsymbol{\alpha})|^{10}\mathrm{d}\boldsymbol{\alpha}.$$

Need to understand contribution from minor arcs.
Let $s \geq 11$, then

$$N(P;\mathfrak{m}) = \int_{\mathfrak{m}} \prod_{i=1}^{s} f(c_i\alpha_2, d_i\alpha_3)\mathrm{d}\boldsymbol{\alpha} \ll \sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})|^{s-10} \int_{[0,1]^2} |f(\boldsymbol{\alpha})|^{10}\mathrm{d}\boldsymbol{\alpha}.$$

Classical: $\sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})| \ll P^{3/4+\varepsilon}$.

Need to understand contribution from minor arcs.
Let $s \geq 11$, then

$$N(P;\mathfrak{m}) = \int_{\mathfrak{m}} \prod_{i=1}^{s} f(c_i\alpha_2, d_i\alpha_3)\mathrm{d}\boldsymbol{\alpha} \ll \sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})|^{s-10} \int_{[0,1]^2} |f(\boldsymbol{\alpha})|^{10}\mathrm{d}\boldsymbol{\alpha}.$$

Classical: $\sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})| \ll P^{3/4+\varepsilon}$.

Theorem (Wooley 2015)

$$\int_{[0,1]^2} |f(\boldsymbol{\alpha})|^{10}\mathrm{d}\boldsymbol{\alpha} \ll P^{5+1/6+\varepsilon}.$$

Need to understand contribution from minor arcs.
Let $s \geq 11$, then

$$N(P;\mathfrak{m}) = \int_{\mathfrak{m}} \prod_{i=1}^{s} f(c_i\alpha_2, d_i\alpha_3)\mathrm{d}\boldsymbol{\alpha} \ll \sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})|^{s-10} \int_{[0,1]^2} |f(\boldsymbol{\alpha})|^{10}\mathrm{d}\boldsymbol{\alpha}.$$

Classical: $\sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})| \ll P^{3/4+\varepsilon}$.

Theorem (Wooley 2015)

$$\int_{[0,1]^2} |f(\boldsymbol{\alpha})|^{10}\mathrm{d}\boldsymbol{\alpha} \ll P^{5+1/6+\varepsilon}.$$

Altogether, when $s \geq 11$:

$$\begin{aligned} N(P) &= N(P;\mathfrak{M}) + O(N(P;\mathfrak{m})) \\ &\sim cP^{s-5} + O(P^{(3/4)(s-10)+\varepsilon} P^{5+1/6+\varepsilon}) \\ &= cP^{s-5} + O(P^{s-5-\delta}). \end{aligned}$$

What happens for larger systems?

We have

$$N(P) = \int_{[0,1]^{r_2+r_3}} \prod_{i=1}^{s} f(\gamma_{i,2}, \gamma_{i,3}) \mathrm{d}\boldsymbol{\alpha},$$

where

$$f(\alpha_2, \alpha_3) = \sum_{x=1}^{P} e(\alpha_2 x^2 + \alpha_3 x^3) \qquad \text{and} \qquad \gamma_{i,k} = \sum_{j=1}^{r_k} c_{i,j} \alpha_{j,k} \qquad (k = 2, 3).$$

What happens for larger systems?

We have

$$N(P) = \int_{[0,1]^{r_2+r_3}} \prod_{i=1}^{s} f(\gamma_{i,2}, \gamma_{i,3}) \mathrm{d}\boldsymbol{\alpha},$$

where

$$f(\alpha_2, \alpha_3) = \sum_{x=1}^{P} e(\alpha_2 x^2 + \alpha_3 x^3) \qquad \text{and} \qquad \gamma_{i,k} = \sum_{j=1}^{r_k} c_{i,j}\alpha_{j,k} \qquad (k = 2, 3).$$

Major arcs dissection as before. Need to understand minor arcs contribution. We have (morally)

$$N(P;\mathfrak{m}) \ll \sup_{\boldsymbol{\alpha}\in\mathfrak{m}} |f(\boldsymbol{\alpha})|^{s-4r_2-6r_3} \int_{[0,1]^{r_2+r_3}} \prod_{i=1}^{r_3} |f(\gamma_{i,2}, \gamma_{i,3})|^{10} \prod_{i=r_3+1}^{r_2} |f(\gamma_{i,2}, \gamma_{i,3})|^{4} \mathrm{d}\boldsymbol{\alpha}.$$

Consider the case $r_2 = 2, r_3 = 1$. The mean value

$$I(P) = \int_{[0,1]^3} |f(\gamma_{1,2}, \gamma_{1,3})|^{10} |f(\gamma_{2,2}, \gamma_{2,3})|^4 \mathrm{d}\boldsymbol{\alpha}$$

counts solutions to the system

$$\begin{aligned} d_{1,1}(x_1^3 + \cdots - y_5^3) \quad &+ d_{1,2}(x_6^3 + \cdots - y_7^3) &= 0 \\ c_{1,1}(x_1^2 + \cdots - y_5^2) \quad &+ c_{1,2}(x_6^2 + \cdots - y_7^2) &= 0 \\ c_{2,1}(x_1^2 + \cdots - y_5^2) \quad &+ c_{2,2}(x_6^2 + \cdots - y_7^2) &= 0. \end{aligned}$$

Consider the case $r_2 = 2, r_3 = 1$. The mean value

$$I(P) = \int_{[0,1]^3} |f(\gamma_{1,2}, \gamma_{1,3})|^{10} |f(\gamma_{2,2}, \gamma_{2,3})|^4 \mathrm{d}\boldsymbol{\alpha}$$

counts solutions to the system

$$\begin{aligned} d_{1,1}(x_1^3 + \cdots - y_5^3) \quad &+ d_{1,2}(x_6^3 + \cdots - y_7^3) &= 0 \\ c_{1,1}(x_1^2 + \cdots - y_5^2) \quad &+ c_{1,2}(x_6^2 + \cdots - y_7^2) &= 0 \\ c_{2,1}(x_1^2 + \cdots - y_5^2) \quad &+ c_{2,2}(x_6^2 + \cdots - y_7^2) &= 0. \end{aligned}$$

We diagonalise the bottom 2×2 matrix:

Consider the case $r_2 = 2, r_3 = 1$. The mean value

$$I(P) = \int_{[0,1]^3} |f(\gamma_{1,2}, \gamma_{1,3})|^{10} |f(\gamma_{2,2}, \gamma_{2,3})|^4 \mathrm{d}\boldsymbol{\alpha}$$

counts solutions to the system

$$\begin{aligned}
d_{1,1}(x_1^3 + \cdots - y_5^3) \quad & + d_{1,2}(x_6^3 + \cdots - y_7^3) && = 0 \\
c_{1,1}(x_1^2 + \cdots - y_5^2) \quad & && = 0 \\
& + c_{2,2}(x_6^2 + \cdots - y_7^2) && = 0.
\end{aligned}$$

Consider the case $r_2 = 2, r_3 = 1$. The mean value

$$I(P) = \int_{[0,1]^3} |f(\gamma_{1,2}, \gamma_{1,3})|^{10} |f(\gamma_{2,2}, \gamma_{2,3})|^4 \mathrm{d}\boldsymbol{\alpha}$$

counts solutions to the system

$$\begin{array}{lll} d_{1,1}(x_1^3 + \cdots - y_5^3) & +d_{1,2}(x_6^3 + \cdots - y_7^3) & = 0 \\ c_{1,1}(x_1^2 + \cdots - y_5^2) & & = 0 \\ & +c_{2,2}(x_6^2 + \cdots - y_7^2) & = 0. \end{array}$$

The last equation is a quadratic form in four variables and has $\ll P^{2+\varepsilon}$ solutions.

Consider the case $r_2 = 2, r_3 = 1$. The mean value

$$I(P) = \int_{[0,1]^3} |f(\gamma_{1,2}, \gamma_{1,3})|^{10} |f(\gamma_{2,2}, \gamma_{2,3})|^4 \mathrm{d}\boldsymbol{\alpha}$$

counts solutions to the system

$$\begin{array}{lll} d_{1,1}(x_1^3 + \cdots - y_5^3) & +d_{1,2}(x_6^3 + \cdots - y_7^3) & = 0 \\ c_{1,1}(x_1^2 + \cdots - y_5^2) & & = 0 \\ & +c_{2,2}(x_6^2 + \cdots - y_7^2) & = 0. \end{array}$$

The last equation is a quadratic form in four variables and has $\ll P^{2+\varepsilon}$ solutions. This fixes the variables x_6, x_7, y_6, y_7, and we need to solve

$$d_{1,1}(x_1^3 + \cdots - y_5^3) = w_1$$
$$c_{1,1}(x_1^2 + \cdots - y_5^2) = 0.$$

Consider the case $r_2 = 2, r_3 = 1$. The mean value

$$I(P) = \int_{[0,1]^3} |f(\gamma_{1,2}, \gamma_{1,3})|^{10} |f(\gamma_{2,2}, \gamma_{2,3})|^4 \mathrm{d}\boldsymbol{\alpha}$$

counts solutions to the system

$$\begin{array}{llr} d_{1,1}(x_1^3 + \cdots - y_5^3) & +d_{1,2}(x_6^3 + \cdots - y_7^3) & = 0 \\ c_{1,1}(x_1^2 + \cdots - y_5^2) & & = 0 \\ & +c_{2,2}(x_6^2 + \cdots - y_7^2) & = 0. \end{array}$$

The last equation is a quadratic form in four variables and has $\ll P^{2+\varepsilon}$ solutions. This fixes the variables x_6, x_7, y_6, y_7, and we need to solve

$$d_{1,1}(x_1^3 + \cdots - y_5^3) = w_1$$
$$c_{1,1}(x_1^2 + \cdots - y_5^2) = 0.$$

This system has $\ll P^{5+1/6+\varepsilon}$ solutions by Wooley's theorem, and we get

$$I(P) \ll P^{2+\varepsilon} P^{5+1/6+\varepsilon} \ll P^{7+1/6+\varepsilon}.$$

The results, part I

This argument leads to

Theorem 1 (JB & S. T. Parsell 2015, to appear)

Suppose that the quadratic and cubic systems are suitably non-singular. We have

$$N(P) \sim cP^{s-2r_2-3r_3},$$

with $c \geq 0$, provided that

- $s \geq 4r_2 + \lfloor (20/3) r_3 \rfloor + 1$ *if* $r_2 \geq r_3$

The results, part I

This argument leads to

Theorem 1 (JB & S. T. Parsell 2015, to appear)

Suppose that the quadratic and cubic systems are suitably non-singular. We have

$$N(P) \sim cP^{s-2r_2-3r_3},$$

with $c \geq 0$, provided that

- $s \geq 4r_2 + \lfloor (20/3) r_3 \rfloor + 1$ *if* $r_2 \geq r_3$,
- $s \geq 8r_3 + \lfloor (8/3) r_2 \rfloor + 1$ *if* $r_3 > r_2$.

The results, part I

This argument leads to

Theorem 1 (JB & S. T. Parsell 2015, to appear)

Suppose that the quadratic and cubic systems are suitably non-singular. We have

$$N(P) \sim cP^{s-2r_2-3r_3},$$

with $c \geq 0$, provided that

- $s \geq 4r_2 + \lfloor (20/3)r_3 \rfloor + 1$ *if* $r_2 \geq r_3$,
- $s \geq 8r_3 + \lfloor (8/3)r_2 \rfloor + 1$ *if* $r_3 > r_2$.

- The argument generalises to arbitrary systems of arbitrary degrees.

The results, part I

This argument leads to

Theorem 1 (JB & S. T. Parsell 2015, to appear)

Suppose that the quadratic and cubic systems are suitably non-singular. We have

$$N(P) \sim cP^{s-2r_2-3r_3},$$

with $c \geq 0$, provided that

- $s \geq 4r_2 + \lfloor (20/3) r_3 \rfloor + 1$ *if* $r_2 \geq r_3$,
- $s \geq 8r_3 + \lfloor (8/3) r_2 \rfloor + 1$ *if* $r_3 > r_2$.

- The argument generalises to arbitrary systems of arbitrary degrees.
- For $r_3 = 1$ we obtain essentially square root cancellation.

The results, part I

This argument leads to

Theorem 1 (JB & S. T. Parsell 2015, to appear)

Suppose that the quadratic and cubic systems are suitably non-singular. We have

$$N(P) \sim cP^{s-2r_2-3r_3},$$

with $c \geq 0$, provided that

- $s \geq 4r_2 + \lfloor (20/3) r_3 \rfloor + 1$ *if* $r_2 \geq r_3$,
- $s \geq 8r_3 + \lfloor (8/3) r_2 \rfloor + 1$ *if* $r_3 > r_2$.

- The argument generalises to arbitrary systems of arbitrary degrees.
- For $r_3 = 1$ we obtain essentially square root cancellation.
- The case $r_3 > r_2$ is clearly unsatisfactory...

The case $r_3 \geq 2r_2$: Complification

Recall

$$N(P) = \int_{[0,1]^{r_2+r_3}} \prod_{i=1}^{s} f(\gamma_{i,2}, \gamma_{i,3}) \mathrm{d}\boldsymbol{\alpha},$$

where

$$f(\alpha_2, \alpha_3) = \sum_{x=1}^{P} e(\alpha_2 x^2 + \alpha_3 x^3) \qquad \text{and} \qquad \gamma_{i,k} = \sum_{j=1}^{r_k} c_{i,j} \alpha_{3,j} \qquad (k = 2, 3).$$

The case $r_3 \geq 2r_2$: Complification

Recall

$$N(P) = \int_{[0,1]^{r_2+r_3}} \prod_{i=1}^{s} f(\gamma_{i,2}, \gamma_{i,3}) \mathrm{d}\boldsymbol{\alpha},$$

where

$$f(\alpha_2, \alpha_3) = \sum_{x=1}^{P} e(\alpha_2 x^2 + \alpha_3 x^3) \qquad \text{and} \qquad \gamma_{i,k} = \sum_{j=1}^{r_k} c_{i,j} \alpha_{3,j} \qquad (k = 2, 3).$$

We have

$$N(P; \mathfrak{m}) \ll \sup_{\boldsymbol{\alpha} \in \mathfrak{m}} |f(\alpha_2, \alpha_3)|^{s-4r_2-6r_3} J_1(P),$$

where

$$J_1(P) = \int_{[0,1]^{r_2+r_3}} \prod_{i=1}^{r_3} |f(\boldsymbol{\gamma}_i)|^2 \prod_{i=r_3+1}^{r_3+r_2} |f(\boldsymbol{\gamma}_i)|^8 \prod_{i=r_3+r_2+1}^{2r_3-2r_2} |f(\boldsymbol{\gamma}_i)|^4 \prod_{i=2r_3-2r_2+1}^{2r_3-r_2} |f(\boldsymbol{\gamma}_i)|^8 \mathrm{d}\boldsymbol{\alpha}$$

Recall

$$J_1(P) = \int_{[0,1]^{r_2+r_3}} \prod_{i=1}^{r_3} |f(\gamma_i)|^2 \prod_{i=r_3+1}^{r_3+r_2} |f(\gamma_i)|^8 \prod_{i=r_3+r_2+1}^{2r_3-2r_2} |f(\gamma_i)|^4 \prod_{i=2r_3-2r_2+1}^{2r_3-r_2} |f(\gamma_i)|^8 \mathrm{d}\boldsymbol{\alpha}.$$

Idea: Do the counter-intuitive thing – blow the system up!

Recall

$$J_1(P) = \int_{[0,1]^{r_2+r_3}} \prod_{i=1}^{r_3} |f(\gamma_i)|^2 \prod_{i=r_3+1}^{r_3+r_2} |f(\gamma_i)|^8 \prod_{i=r_3+r_2+1}^{2r_3-2r_2} |f(\gamma_i)|^4 \prod_{i=2r_3-2r_2+1}^{2r_3-r_2} |f(\gamma_i)|^8 \mathrm{d}\boldsymbol{\alpha}.$$

Idea: Do the counter-intuitive thing – blow the system up!

Cauchy-Schwarz gives

$$J_n(P) \ll (P^{(5+1/6)r_2+\varepsilon})^{1/2} J_{2n}(P)^{1/2},$$

where the cubic system is of the shape

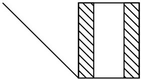

Figure: $J_1(P)$

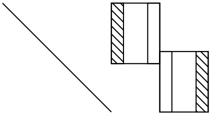

Figure: $J_2(P)$

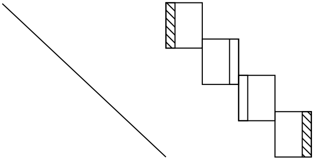

Figure: $J_4(P)$

After m inductive steps:

$$J_1(P) \ll \left(P^{(5+1/6)r_2+\varepsilon}\right)^{1-2^{-m}} J_{2^m}(P)^{2^{-m}}$$

The mean value $J_n(P)$ contains $R_n = n(r_3 - r_2) + r_2$ cubic and r_2 quadratic polynomials in $6R_n + 4r_2$ variables.

After m inductive steps:

$$J_1(P) \ll (P^{(5+1/6)r_2+\varepsilon})^{1-2^{-m}} J_{2^m}(P)^{2^{-m}}$$

The mean value $J_n(P)$ contains $R_n = n(r_3 - r_2) + r_2$ cubic and r_2 quadratic polynomials in $6R_n + 4r_2$ variables.

Now a miracle happens:

$$J_n(P) \ll P^{3n(r_3-r_2)+9r_2+\varepsilon} \ll P^{3R_n+6r_2+\varepsilon}.$$

We miss by a constant exponent – regardless of the value of n!!!

After m inductive steps:

$$J_1(P) \ll (P^{(5+1/6)r_2+\varepsilon})^{1-2^{-m}} J_{2^m}(P)^{2^{-m}}$$

The mean value $J_n(P)$ contains $R_n = n(r_3 - r_2) + r_2$ cubic and r_2 quadratic polynomials in $6R_n + 4r_2$ variables.

Now a miracle happens:

$$J_n(P) \ll P^{3n(r_3-r_2)+9r_2+\varepsilon} \ll P^{3R_n+6r_2+\varepsilon}.$$

We miss by a constant exponent – regardless of the value of n!!!

This gives

$$J_1(P) \ll (P^{(5+1/6)r_2+\varepsilon})^{1-2^{-m}} (P^{3\cdot 2^m(r_3-r_2)+9r_2+\varepsilon})^{2^{-m}}$$
$$\ll P^{3r_3+(2+1/6)r_2+2^{-m}\frac{23}{6}r_2+\varepsilon}.$$

The results, part II

This leads to

Theorem 2 (JB 2016, to appear)

Suppose that the quadratic and cubic systems are suitably non-singular, and $r_3 \geq 2r_2 > 1$. *We have*

$$N(P) \sim cP^{s-2r_2-3r_3}$$

with $c \geq 0$*, provided that*

$$s \geq 6r_3 + \lfloor (14/3)r_2 \rfloor + 1.$$

The results, part II

This leads to

Theorem 2 (JB 2016, to appear)

Suppose that the quadratic and cubic systems are suitably non-singular, and $r_3 \geq 2r_2 > 1$. *We have*

$$N(P) \sim cP^{s-2r_2-3r_3}$$

with $c \geq 0$, *provided that*

$$s \geq 6r_3 + \lfloor (14/3)r_2 \rfloor + 1.$$

- For $r_3 \notin (r_2, 2r_2)$, we have the bound

$$s \geq 4r_2 + 6r_3 + \lfloor (2/3)\min\{r_2, r_3\} \rfloor + 1.$$

Unfortunately, the middle range for r_3 seems to be harder.

The results, part II

This leads to

Theorem 2 (JB 2016, to appear)

Suppose that the quadratic and cubic systems are suitably non-singular, and $r_3 \geq 2r_2 > 1$. *We have*

$$N(P) \sim cP^{s-2r_2-3r_3}$$

with $c \geq 0$, *provided that*

$$s \geq 6r_3 + \lfloor (14/3)r_2 \rfloor + 1.$$

- For $r_3 \notin (r_2, 2r_2)$, we have the bound

  $$s \geq 4r_2 + 6r_3 + \lfloor (2/3)\min\{r_2, r_3\} \rfloor + 1.$$

  Unfortunately, the middle range for r_3 seems to be harder.
- We have square root cancellation for $\min\{r_2, r_3\} = 1$.

The end

Thank you for your attention!