ROOT NUMBER IN FAMILIES OF ELLIPTIC CURVES

$E : y^2 = x^3 + ax + b, a, b \in \mathbb{Q}$, $w(E)$ = root number is defined as follows:

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

$$\Lambda(s, E) = \left(\frac{\sqrt{N_E}}{2\pi}\right)^s \Gamma(s) L(s, E) = w(E)\Lambda(2 - s, E).$$

$w(E) \in \{\pm 1\}$, and $w(E) = -1 \implies L(1, E) = 0$.

Differentiate $w(E) = -1 \Leftrightarrow ord_{s=1} L(s, E) = odd$.

$$w(E) = (-1)^{ord_{s=1} L(s,E)} \underset{\substack{\uparrow \\ \text{parity conjecture} \\ (\Longleftarrow BSD)}}{=} (-1)^{rk(E)}$$

Conjecture:

$$Av(w(E(a, b))) = 0$$

$$Av(rk(E(a, b))) = \frac{1}{2}$$

What about one parameter families?

$$\mathcal{E} : y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t),\ a_i(t) \in \mathbb{Z}[t].$$

For "most" $\mathcal{E}$, $Av_{\mathbb{Z}}(w(\mathcal{E}(t))) = 0$ and $Av_{\mathbb{Q}}(w(\mathcal{E}(t))) = 0$.

$$Av_{\mathbb{Z}}(w_{\mathcal{E}}) = \lim_{T\to\infty} \frac{1}{2T} \sum_{|t|\le T, t\in\mathbb{Z}} w(\mathcal{E}(t))$$

$$Av_{\mathbb{Q}}(w_{\mathcal{E}}) = \lim_{X\to\infty} \frac{1}{\#\{t \in \mathbb{Q}, h(t) \le X\}} \sum_{t\in\mathbb{Q}, h(t)\le X} w(\mathcal{E}(t))$$

Theorem 1 (Rizzo)**.**

1. $\mathcal{F} : y^2 = x^3 + tx^2 - (t + 3)x + 1, rk = 1/\mathbb{Q}(t), w(\mathcal{F}(t)) = -1, \forall t \in \mathbb{Z}$.
2. $\mathcal{E} : y^2 = x^3 + \frac{t}{4}x - \frac{36t^2}{t - 1728}x + \frac{t^3}{t - 1728}, j(t) = t, Av_{\mathbb{Z}}(w(\mathcal{E}(t))) = 0.003\ldots.$

Other examples: Romano, Helfgott $Av_{\mathbb{Q}} \notin \{-1, 0, 1\}$

Applications: one level density in family $\mathcal{F}$.

Want:

(1) Find families with non zero $Av_{\mathbb{Z}}$, $Av_{\mathbb{Q}}$.

(2) Can we have excess rank?

(3) Can we get $Av_{\mathbb{Z}} = \frac{h}{k}$ (or $Av_{\mathbb{Q}} = \frac{h}{k}$) for a given $\frac{h}{k}$?

How do we compute $w(E), E/\mathbb{Q}$?

$$w(E) = \prod_{p|N_E} w_p(E),$$

where $w_p(E)$ are local root numbers which can be computed and recorded in tables of Halberstadt, Connell and Rohrlich. For $p \geq 5$,

$v_p(c_4, c_6, \Delta)$	$v_p(N_E)$	$w_p(E)$
$(0, 0, \geq 0)$	0	1
$(0, 0, \geq 1)$	1	$-\left(\frac{-(c_6)_p}{p}\right)$
9 cases	2	$\left(\frac{-1}{p}\right), \left(\frac{-2}{p}\right)$ or $\left(\frac{-3}{p}\right)$

where $x = p^{v_p(x)} x_p$.

Theorem 2 (Helfgott). *Let $\mathcal{F}$ be a family of elliptic curves, then*

$$w(\mathcal{E}(t)) = sgn(g_\infty(t))\lambda(\mathcal{M}_{\mathcal{F}}(t)) \prod_p g_p(t),$$

where

$$\mathcal{M}_{\mathcal{F}}(t) = \prod_{P(t) mult} P(t), \quad \mathcal{B}_{\mathcal{F}}(t) = \prod_{P(t) quite\ bad} P(t),$$

$g_p(t)$ are p-adic locally constant and $g_p(t) = 1$ for $v_p(\mathcal{B}_{\mathcal{F}}(t)) = 0, 1, \forall p \notin S$(finite).Furthermore, assuming Chowla's conjecture and square free sieve conjecture, if $\mathcal{M}_{\mathcal{F}}(t) \neq 1$, then$Av_{\mathbb{Z}}(w(\mathcal{F}(t))) = 0$. If $\mathcal{M}_{\mathcal{F}}(t) = 1$, assuming SF conjecture , then

$$Av(w_{\mathcal{F}}(t)) = \frac{c_+ c_+}{2} \prod_p \int_{\mathbb{Z}_p} g_p(t) dt$$

where $c_\pm = \lim_{x \to \pm\infty} sgn(g_\infty(t))$

Definition:

(1) $\mathcal{F}$ is potentially parity biased if $\mathcal{M}_{\mathcal{F}}(t) = 1$.
(2) $\mathcal{F}$ is parity biased if $Av_{\mathbb{Z}}(\mathcal{F}(t)) \neq 0$.
(3) $\mathcal{F}$ has excess rank if $Av_{\mathbb{Z}} = -(-1)^{rk(\mathcal{F})/\mathbb{Q}(t)}$.

Theorem 3 (Bettin, David & Delaunay). *We can classify all potentially biased $\mathcal{F}$ s.t. $\deg a_i(t) \leq 2$, which fall in 6 families of families.*

- $\mathcal{E}_s : y^2 = x^3 + 3tx^3 + 3sx + st, s \in \mathbb{Z}_{\neq 0}$.
- $\mathcal{G}_w : wy^2 = x^3 + 3tx^3 + 3tx + t^2, w \in \mathbb{Z}_{\neq 0}$
- $\mathcal{H}_w, \mathcal{I}_w, \mathcal{J}_{m,w}, \mathcal{L}_{m,s,v}$

all of whose rank/$\mathbb{Q}(t)$ are $0, 1$ except $\mathcal{L}_{m,s,v}$ whose rank are $0, 1, 2, 3$.

Example:

$$\mathcal{F}_a : y^2 = x^3 + tx^2 - a(t+3a)x + a^3, a \in \mathbb{Z}_{\neq 0},$$

$\Delta(t) = 16a^2 f_a^2(t), f_a(t) = t^2 + 3at + 9a^2$, $rk = 0(a \neq \pm\square), 1(a = \pm\square)$

$$\begin{aligned}
w(\mathcal{F}_a(t)) &= -w_2(t) \prod_{\substack{p \geq 3 \\ 0 \leq v_p(a) \leq v_p(t)}} \left(\frac{-1}{p}\right)^{v_p(a)+v_p(f_a(t))} \prod_{\substack{p \geq 3 \\ 0 \leq v_p(t) \leq v_p(a)}} \begin{bmatrix} \left(\frac{-t_p}{p}\right) & v_p(t) \text{ even} \\ \left(\frac{-1}{p}\right) & v_p(t) \text{ odd} \end{bmatrix} \\
&= -w_2(t) \prod_{p \geq 3} \left(\frac{-1}{p}\right)^{v_p(a)+v_p(f_a(t))} \prod_{\substack{p \geq 3 \\ 0 \leq v_p(t) \leq v_p(a)}} \left(\frac{-t_p}{p}\right)^{1+v_p(t)} \left(\frac{-1}{p}\right)^{v_p(a)+v_p(t)} \\
&\equiv -w_2(t)(af_a(t))_2 \mathrm{sgn}((af_a(t))) \prod_{p|a} w_p^*(t) \pmod 4 \\
&\equiv \mathrm{sgn}(af_a(t)) w_2^*(t) \prod_{p|a} w_a^*(t) \pmod 4
\end{aligned}$$

$$\implies Av_{\mathbb{Z}}(\mathcal{F}_a(t)) = \mathrm{sgn}(a) \prod_{p|2a} \int w_p^*(t) dt$$

Corollary 1. *If a is odd and square free,*

$$Av_{\mathbb{Z}} = \pm\frac{1}{a}, \pm\frac{1}{2a},$$

depanding on $a \pmod 8$.

Corollary 2. *If a is a prime and $a \equiv \pm 1 \pmod 8$, b is a non residue $\pmod a$. Then $\mathcal{F}_a(at+b)$ has rank 0 and $w(\mathcal{F}_a(t)) = -1 \forall t$. $\mathcal{F}_{a^2}(at+b^2)$ has rank 1 and $w(\mathcal{F}_{a^2}(t)) = -1 \forall t$.*

Density result:

Theorem 4 (Rizzo (isotrivial families))**.** *$Av_{\mathbb{Q}}(w(\mathcal{E}(t)))$ is dense in $[-1,1]$ when $\mathcal{E}$ varies over the twists by $f(t)$ of $\mathcal{E}/\mathbb{Q}$.*

Theorem 5 (Bettin, David & Delaunay)**.**

$$Av_{\mathbb{Z}}(w(\mathcal{E}(t))) \supset [-1,1] \cap \mathbb{Q},$$

$$Av_{\mathbb{Q}}(w(\mathcal{E}(t))) \textit{ is dense in } [-1,1],$$

where $\mathcal{E}$ varies over non-isotrivial families.

0.1. $Av_{\mathbb{Z}}$**.** In fact, fix $\frac{h}{k}$, can build $\mathcal{E}$ s.t. $Av_{\mathbb{Z}}(w(\mathcal{E}(t))) = \frac{h}{k}$. $\mathcal{E}(t) = \mathcal{F}_{a(t)}(Q(t))$, where $a(t)$ and $Q(t)$ depend on h and k and such that the root number is localized at 1 prime.

$$w(\mathcal{E}(t)) = (-1)^{1+v_p(Q(t))} \left(\frac{t_p}{p}\right)^{1+v_p(Q(t))}, \ p \equiv -1 \pmod{2k},$$

where $a(t) = 2^4 p^{P(t)}$, $Q(t) = (4pt^2+1)P(t)$, $P(t) = -p\prod_{i=1}^{m}(t-i)$, $m = p+1-2rh, p+1 = 2kr$.

0.2. $Av_{\mathbb{Q}}$**.** Trick: the root number will be the variation of the sign of a polynomial $\mathcal{F}_{a_x(t)}(Q_x(t))$ as $x \to \infty$.

Theorem 6. *Let $w(r,s) = w_\infty(r,s)\prod_p w_p(r,s)$, where $w_p(r,s)$ are p-adic locally constant and $w(r,s) = 1$ when $v_p(B(r,s)) = 0, 1$ for some $B(r,s) \in \mathbb{Z}[r,s]$, $\forall p \notin S$, a finite set. Then*

$$Av_{\mathbb{Z}^2,coprime}(w(r,s)) = c_\infty \prod_p \frac{1}{1-p^{-2}} \int_{\mathbb{Z}_p\times\mathbb{Z}_p\setminus p\mathbb{Z}_p\times p\mathbb{Z}_p} w_p(r,s)drds,$$

where $c_\infty = \frac{1}{4N^2}\int_{-N}^{N}\int_{-N}^{N} g_\infty(r,s)drds$, $g_\infty(r,s) =$ sgn of a polynomial of 2 variables such that $c_\infty = \frac{h}{k}$.