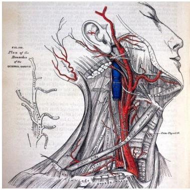
Sums of distinct divisors

Lola Thompson

Oberlin College

February 3, 2017

The "anatomy" of integers



What does it mean to study the "anatomy" of integers? Some natural problems/goals:

- Study the prime factors of integers, their size and their quantity.
- Obtain good bounds for the number of integers with certain properties (e.g., those with only large prime factors).
- Understand the distribution of divisors of integers in a given interval.

Integers with dense divisors

Several papers in this area study the set of integers with z-dense divisors.

Let $1 = d_1(n) < d_2(n) < \cdots < d_{\tau(n)}(n) = n$ denote the sequence of divisors of an integer n.

Definition

For $z \geq 2$, an integer n is **z-dense** if $\max_{1 \leq i < \tau(n)} \frac{d_{i+1}(n)}{d_i(n)} \leq z$.

Example. 6 is 2-dense, since $\frac{2}{1}, \frac{3}{2}, \frac{6}{3} \leq 2$.

Integers with dense divisors

$$F(n) := \begin{cases} 1 & n = 1 \\ \max\{dP^-(d) : d \mid n, d > 1\} & n \geq 2. \end{cases}$$

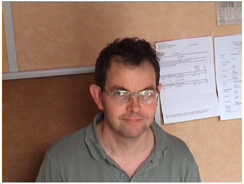
Theorem (Tenenbaum)

For $n \geq 2$,

$$\frac{F(n)}{n} = \max_{1 \leq i < \tau(n)} \frac{d_{i+1}(n)}{d_i(n)}.$$

So n has z-dense divisors if $F(n) \leq nz$.

Integers with dense divisors



Let $D(x,z) = \#\{n \le x : n \text{ is } z\text{-dense}\}$.

Tenenbaum obtained upper and lower bounds for $D(x,z)$, which were later improved by Saias to

$$D(x,z) \asymp \frac{x \log z}{\log x}.$$

Proof sketch

For any fixed $z \geq 2$, let $\mathcal{D}(x) = \{n \leq x : F(n) \leq nz\}$.

Let $D(x, y) = \#\{n \in \mathcal{D}(x) : P^+(n) \leq y\}$.

Then

$$D(x, y) = 1 + \sum_{p \leq \min(y, h(x))} D(x/p, p) + ET.$$

(E.g., Can take $h(x) = \sqrt{x}$ so that ET is negligible.)

Smoothed version: $D^*(x, y) = \int_1^{\min(y, \sqrt{x})} D^*(x/t, t) \frac{dt}{\log t}$.

Can *almost* take $D^*(x, y) = x\rho(u-1)/\log x$, where ρ is defined by $u\rho'(u) + \rho(u-1) = 0$ and $\rho(u) = 1$ for $u \leq 1$.

Two natural applications

Our talk will focus on two natural applications of Tenenbaum's work on integers with z-dense divisors:

1. How often is it the case that every m in $[1, n]$ can be written as a sum of distinct divisors of n?

Two natural applications

Our talk will focus on two natural applications of Tenenbaum's work on integers with z-dense divisors:

1. How often is it the case that every m in $[1, n]$ can be written as a sum of distinct divisors of n?

2. How often does the polynomial $x^n - 1$ have a divisor of every degree between 1 and n in $\mathbb{Z}[x]$?

Practical numbers

Definition

A positive integer n is **practical** if every m with $1 \leq m \leq \sigma(n)$ can be written as a sum of distinct divisors of n.

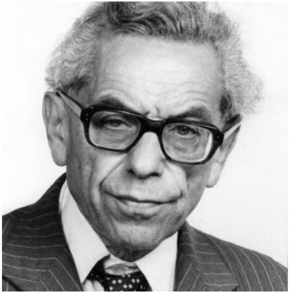
Example. $n = 6$

Divisors: $1, 2, 3, 6$

Nonexample. $n = 10$

Divisors: $1, 2, 5, 10$

Practical numbers: a short history



Theorem (Erdős, 1950)

Let $PR(X) := \#\{n \leq X : n \text{ is practical}\}$. *Then*

$$PR(X) = o(X).$$

Practical numbers: a short history

Hausman and Shapiro, 1983:

$$PR(X) \ll \frac{X}{(\log X)^{\beta}}, \qquad \beta = \frac{(1-1/\log 2)^2}{2} = 0.0979\ldots.$$

Margenstern, 1984:

$$PR(X) \gg \frac{X}{\exp(\alpha(\log\log X)^2)}, \qquad \alpha = \frac{1+\varepsilon}{2\log 2} = 0.7213\ldots.$$

Tenenbaum, 1986:

$$\frac{X}{(\log X)(\log\log X)^{4.21}} \ll PR(X) \ll \frac{X(\log\log X)(\log\log\log X)}{\log X}.$$

Practical numbers: a short history

Theorem (Saias, 1997)

For all $X \geq 2$,

$$PR(X) \asymp \frac{X}{\log X}.$$

Practical vs. φ-Practical

Definition

A positive integer n is φ-**practical** if every m with $1 \leq m \leq n$ can be written as $\sum_{d \in \mathcal{D}} \varphi(d)$, where $\mathcal{D}$ is a subset of divisors of n.

Note: Since $x^n - 1 = \prod_{d|n} \Phi_d(x)$, this is equivalent to the condition that $x^n - 1$ has at least one divisor of every degree between 1 and n.

φ-practical example

Example. $n = 6$

Divisors: $1, 2, 3, 6$

φ values: $1, 1, 2, 2$

Nonexample. $n = 66$ is practical but **not** φ-practical.

Divisors: $1, 2, 3, 6, 11, 22, 33, 66$

φ values: $1, 1, 2, 2, 10, 10, 22, 22$

φ-practical example

Example. $n = 6$

Divisors: $1, 2, 3, 6$

φ values: $1, 1, 2, 2$

Nonexample. $n = 66$ is practical but **not** φ-practical.

Divisors: $1, 2, 3, 6, 11, 22, 33, 66$

φ values: $1, 1, 2, 2, 10, 10, 22, 22$

Exercise. Every even φ-practical is practical.

Counting the number of φ-practicals

We can prove the following analogue of Saias' result for the φ-practical numbers:

Theorem (T., 2013)

Let $F(X) = \#\{n \leq X : n$ *is* φ*-practical*$\}$*. Then*

$$F(X) \asymp \frac{X}{\log X}.$$

A key obstruction

The proofs of Saias et al. relied heavily on the following:

Theorem (Stewart, 1954)

Let $n = p_1^{e_1} \cdots p_j^{e_j}$, $n > 1$, *with* $p_1 < p_2 < \cdots < p_j$ *prime and* $e_i \geq 1$ *for* $i = 1, \ldots, j$. *Then* n *is practical iff for all* $i = 1, \ldots, j$, $p_i \leq \sigma(p_1^{e_1} \cdots p_{i-1}^{e_{i-1}}) + 1$.

Unfortunately, there's no simple method for building up φ-practical numbers from smaller ones.

Example. $3^2 \times 5 \times 17 \times 257 \times 65537 \times (2^{31} - 1)$ is φ-practical, but none of the numbers 3^2, $3^2 \times 5$, $3^2 \times 5 \times 17$, $3^2 \times 5 \times 17 \times 257$, $3^2 \times 5 \times 17 \times 257 \times 65537$ are φ-practical.

Proof of the upper bound

Instead, we devise the following workaround:

Definition

Let $n = p_1^{e_1} \cdots p_k^{e_k}$. Let $m_i = p_1^{e_1} \cdots p_i^{e_i}$. We define an integer n to be *weakly φ-practical* if the inequality $p_{i+1} \le m_i + 2$ holds for all i.

Lemma

Every φ-practical number is weakly φ-practical.

Note: The converse does **not** hold. For example, 45 is not φ-practical but it is weakly φ-practical.

Proof of the upper bound

To prove our theorem, we consider two cases:

- If n is **even** & φ-practical then $p_{i+1} \leq m_i + 2 \leq \sigma(m_i) + 1$ for all $i \geq 1$. Hence, each m_i satisfies the inequality in Stewart's Condition, so n is practical.

Proof of the upper bound

To prove our theorem, we consider two cases:

- If n is **even** & φ-practical then $p_{i+1} \leq m_i + 2 \leq \sigma(m_i) + 1$ for all $i \geq 1$. Hence, each m_i satisfies the inequality in Stewart's Condition, so n is practical.
- On the other hand, observe that for every $n \in (0, X]$, there is a unique k such that $2^k n \in (X, 2X]$. Then, if n is **odd** & φ-practical, $2^k n$ will be practical.

Proof of the upper bound

To prove our theorem, we consider two cases:

- If n is **even** & φ-practical then $p_{i+1} \leq m_i + 2 \leq \sigma(m_i) + 1$ for all $i \geq 1$. Hence, each m_i satisfies the inequality in Stewart's Condition, so n is practical.
- On the other hand, observe that for every $n \in (0, X]$, there is a unique k such that $2^k n \in (X, 2X]$. Then, if n is **odd** & φ-practical, $2^k n$ will be practical.
- Thus, $F(X) \leq PR(2X) \ll \frac{X}{\log X}$, by Saias' Theorem.

Lower Bound Proof Sketch

Saias obtains his lower bound by comparing the set of practical numbers with the set of integers with 2-dense divisors:

Definition

An integer n is 2-*dense* if $\max_{1 \leq i \leq \tau(n)-1} \frac{d_{i+1}(n)}{d_i(n)} \leq 2$.

Note: All integers with 2-dense divisors are practical, but the same cannot be said about the φ-practical numbers. For example, $n = 66$ is 2-dense but it is not φ-practical.

Lower Bound Proof Sketch

We obtain our lower bound by comparing the set of φ-practical numbers with the set of integers with strictly 2-dense divisors:

Definition

An integer n is *strictly 2-dense* if $\max_{1<i<\tau(n)-1} \frac{d_{i+1}(n)}{d_i(n)} < 2$ and $\frac{d_2(n)}{d_1(n)} = 2 = \frac{d_{\tau(n)}(n)}{d_{\tau(n)-1}(n)}$.

It turns out that all strictly 2-dense integers are φ-practical.

Lower Bound Proof Sketch

Goal: Show that a positive proportion of 2-dense integers are strictly 2-dense, except for some possible obstructions at small primes.

Lower Bound Proof Sketch

Goal: Show that a positive proportion of 2-dense integers are strictly 2-dense, except for some possible obstructions at small primes.

1. First find an upper bound for the number of integers up to X that are 2-dense but not strictly 2-dense:

$$\sum_{k>C} \sum_{\substack{m\in(2^{k-1},2^k) \\ m \text{ 2-dense}}} \sum_{\substack{p\in(2^{k-1},2^{k+1}) \\ p \text{ prime}}} \sum_{\substack{j\leq X/mp \\ mpj \text{ 2-dense} \\ P^-(j)>p}} 1.$$

Lower Bound Proof Sketch

Goal: Show that a positive proportion of 2-dense integers are strictly 2-dense, except for some possible obstructions at small primes.

1. First find an upper bound for the number of integers up to X that are 2-dense but not strictly 2-dense:

$$\sum_{k>C} \sum_{\substack{m\in(2^{k-1},2^k) \\ m \text{ 2-dense}}} \sum_{\substack{p\in(2^{k-1},2^{k+1}) \\ p \text{ prime}}} \sum_{\substack{j\leq X/mp \\ mpj \text{ 2-dense} \\ P^-(j)>p}} 1.$$

2. Use Brun's sieve and other classical techniques from multiplicative number theory to show that the number counted above is $\leq \varepsilon\frac{X}{\log X}$.

Lower Bound Proof Sketch

- Show that a subset of the strictly 2-dense integers is in one-to-one correspondence with a positive proportion of the 2-dense integers with obstructions at $k < C$.

Corollary (T., 2013)

For X sufficiently large, we have

$$\#\{n \leq X : n \text{ is practical but not } \varphi\text{-practical}\} \gg \frac{X}{\log X}.$$

Moreover, we also have

$$\#\{n \leq X : n \text{ is } \varphi\text{-practical but not practical}\} \gg \frac{X}{\log X}.$$

An asymptotic for the practicals

Theorem (Weingartner, 2015)

There exists a positive constant c *such that for* $X \geq 3$,

$$PR(X) = \frac{cX}{\log X}\left(1 + O\left(\frac{\log\log X}{\log X}\right)\right).$$

Proof sketch

Idea: Count all integers $n \leq x$ according to their practical part:

$$n = (p_1^{e_1} \cdots p_j^{e_j})(p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}) = m \cdot r$$

where $m = p_1^{e_1} \cdots p_j^{e_j}$ is **practical** but $p_1^{e_1} \cdots p_j^{e_j} p_{j+1}^{e_{j+1}}$ is **not**.

Proof sketch

Idea: Count all integers $n \le x$ according to their practical part:

$$n = (p_1^{e_1} \cdots p_j^{e_j})(p_{j+1}^{e_{j+1}} \cdots p_k^{e_k}) = m \cdot r$$

where $m = p_1^{e_1} \cdots p_j^{e_j}$ is **practical** but $p_1^{e_1} \cdots p_j^{e_j} p_{j+1}^{e_{j+1}}$ is **not**.

Then

$$\lfloor x \rfloor = \sum_{\substack{m \le x \\ m \text{ practical}}} \Phi(x/m, \sigma(m) + 1)$$

where $\Phi(x, y) = \#\{n \le x : p \mid n \Rightarrow p > y\}$.

Proof sketch

Since

- $$\lfloor x \rfloor = \sum_{\substack{m \le x \\ m \text{ practical}}} \Phi(x/m, \sigma(m) + 1)$$

Proof sketch

Since

- $$\lfloor x \rfloor = \sum_{\substack{m \leq x \\ m \text{ practical}}} \Phi(x/m, \sigma(m)+1)$$
- $$1 = \sum_{m \text{ practical}} \frac{1}{m} \prod_{p \leq \sigma(m)+1} \left(1 - \frac{1}{p}\right)$$

Proof sketch

Since

- $$\lfloor x \rfloor = \sum_{\substack{m \le x \\ m \text{ practical}}} \Phi(x/m, \sigma(m)+1)$$
- $$1 = \sum_{m \text{ practical}} \frac{1}{m} \prod_{p \le \sigma(m)+1} \left(1 - \frac{1}{p}\right)$$

we have

$$0 = \sum_{m \text{ practical}} \left(\Phi(x/m, \sigma(m)+1) - \frac{\lfloor x \rfloor}{m} \prod_{p \le \sigma(m)+1} \left(1 - \frac{1}{p}\right) \right)$$

Proof sketch

$$0 = \sum_{m \text{ practical}} \left(\Phi(x/m, \sigma(m)+1) - \frac{\lfloor x \rfloor}{m} \prod_{p \le \sigma(m)+1} \left(1 - \frac{1}{p}\right) \right)$$

Observe that:

Proof sketch

$$0 = \sum_{m \text{ practical}} \left(\Phi(x/m, \sigma(m)+1) - \frac{\lfloor x \rfloor}{m} \prod_{p \le \sigma(m)+1} \left(1 - \frac{1}{p}\right) \right)$$

Observe that:

- $\Phi(x,y) \approx e^{\gamma} x \omega\left(\frac{\log x}{\log y}\right) \prod_{p \le y} \left(1 - \frac{1}{p}\right)$
- $\prod_{p \le y} \left(1 - \frac{1}{p}\right) \approx \frac{e^{-\gamma}}{\log y}$
- $\log(\sigma(m)+1) \approx \log(2m)$

Proof sketch

$$0 = \sum_{m \text{ practical}} \left(\Phi(x/m, \sigma(m)+1) - \frac{\lfloor x \rfloor}{m} \prod_{p \le \sigma(m)+1} \left(1 - \frac{1}{p}\right) \right)$$

Observe that:

- $\Phi(x, y) \approx e^{\gamma} x \omega\left(\frac{\log x}{\log y}\right) \prod_{p \le y} \left(1 - \frac{1}{p}\right)$
- $\prod_{p \le y} \left(1 - \frac{1}{p}\right) \approx \frac{e^{-\gamma}}{\log y}$
- $\log(\sigma(m)+1) \approx \log(2m)$

Use partial summation, get an integral equation, apply a Laplace transform, and invert the Laplace transform.

An asymptotic for the φ-practicals

Theorem (Pomerance, T., Weingartner, 2016)

There exists a positive constant C such that for $X \geq 2$,

$$F(X) = \frac{CX}{\log X}\left(1 + O\left(\frac{1}{\log X}\right)\right).$$

Proof Sketch: Starters

Definition

A **starter** is a φ-practical number m such that either $m/P^+(m)$ is not φ-practical or $P^+(m)^2 \mid m$.

Note: A φ-practical number n is said to have starter m if m is a starter, m is an initial divisor of n, and n/m is squarefree.

Examples:

- 4 is the only starter with squarefull part 4.
- There are only 3 starters with squarefull part 49: $294 = 2 \cdot 3 \cdot 7^2$, $1470 = 2 \cdot 3 \cdot 5 \cdot 7^2$, $735 = 3 \cdot 5 \cdot 7^2$.
- There are infinitely many starters with squarefull part 9.

Proof Sketch:

1. Partition φ-practicals according to their starters: $n = mb$.

2. Use Weingartner's machinery to show that, for any fixed $m \geq 1$, there exist sequences of real numbers c_m and r_m such that

$$B_m(x) = c_m \frac{x}{\log x} + O\left(r_m \frac{x}{\log^2 x}\right),$$

where $B_m := \#\{\varphi\text{-practical } n \leq x \text{ with starter } m\}$.

3. Show that $\sum c_m$ and $\sum r_m$ are finite.

Estimating the asymptotic constant

We can use Sage to compute $F(X)/\frac{X}{\log X}$:

X	$F(X)$	$F(X)/(X/\log X)$	$F(X)/(\mathrm{Li}(X))$
10^1	6	1.381551	0.973141
10^2	28	1.289448	0.929425
10^3	174	1.201949	0.979676
10^4	1198	1.103399	0.961371
10^5	9301	1.070817	0.965855
10^6	74461	1.028717	0.947009
10^7	635528	1.024350	0.955799
10^8	5525973	1.017922	0.959002
10^9	48386047	1.002717	0.951559
10^{10}	431320394	0.993152	0.947841
10^{11}	3907994621	0.989834	0.948988

Table: Ratios for φ-practicals

A generalization

Let

$$S_f(n) = \sum_{d|n} f(d) = f * \mathbb{1}(n).$$

Definition

A positive integer n is called f-**practical** if for every positive integer $m \leq S_f(n)$ there is a set $\mathcal{D}$ of divisors of n for which

$$m = \sum_{d \in \mathcal{D}} f(d)$$

holds.

Example $f = I$: practical numbers.

Example $f = \varphi$: φ-practical numbers.

Densities of f-practicals

Example All positive integers are τ-practical.

Densities of f-practicals

Example All positive integers are τ-practical.

Example The set of λ-practical numbers has asymptotic density 0.

Densities of f-practicals

Example All positive integers are τ-practical.

Example The set of λ-practical numbers has asymptotic density 0.

Example Let $g : \mathbb{N} \to \mathbb{N}$, where $g(1) = 1$, $g(2^k) = 2$, and $g(p^k) = 3$ for all $p \geq 3$ and all $k \geq 1$. The set of g-practical numbers has asymptotic density $1/2$.

Densities of f-practicals

Theorem (Schwab, T., 2016)

For each $n \in \mathbb{N}$, there is a function f_n such that the asymptotic density of f_n-practical numbers in $\mathbb{N}$ is $1 - \frac{\varphi(n)}{n}$.

Densities of f-practicals

Theorem (Schwab, T., 2016)

For each $n \in \mathbb{N}$, there is a function f_n such that the asymptotic density of f_n-practical numbers in $\mathbb{N}$ is $1 - \frac{\varphi(n)}{n}$.

Corollary (Schwab, T., 2016)

The densities of f-practical sets are dense in $[0, 1]$.

Other results

We also:

- Classified the multiplicative functions f for which the f-practical numbers can be completely determined via a Stewart-like criterion;

Other results

We also:

- Classified the multiplicative functions f for which the f-practical numbers can be completely determined via a Stewart-like criterion;
- Proved Chebyshev-type bounds for certain f-practical sets;

Other results

We also:

- Classified the multiplicative functions f for which the f-practical numbers can be completely determined via a Stewart-like criterion;
- Proved Chebyshev-type bounds for certain f-practical sets;
- Classified the additive functions f for which all positive integers are f-practical.

Thank you!