INTRODUCTORY TALK (TARGETED IN PARTICULAR TO MEMBERS TO THE HARMONIC ANALYSIS PROGRAM)

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Finding integer of a special shape:
- prove existence
- determine if there are infinitely many such integers
- if so, "count" them

- Primes, Euclid $\rightsquigarrow$ Riemann hypothesis
- Primes in arithmetic progressions
$$(q, a) \qquad p = qk + a, \quad k \in \mathbb{N}$$
Dirichlet $\rightsquigarrow$ GRH
- twin primes: $p' = p + 2$
Goldbach problem: $2n \stackrel{?}{=} p + p'$
Y. Zhang, Polymath, Maymard, Tao
There exist infinitely many pairs $p < p'$ such that $p - p' \leqslant 246$.
- Matomaki-Radziwill-Tao
Noncorrelation of the Liouville function under additive shifts with itself
- sums of three primes: $2n + 1 = p + p' + p''$
Vinogradov, Helfgott
- linear equations in the primes: Green-Tao
- $x^p + y^p + z^p = 0$ with $(x, y, x, p) \in \mathbb{I}^3 \times \mathcal{P}$
- Given n (large enough), $x_1^\ell + x_2^\ell + \cdots + x_k^\ell = n$, $\ell \geqslant 2$ (Waring problem)
OK if $k \geqslant k(\ell)$ Circle method, Wooley, Bourgan, Demeter, Guth

Existence: necessary conditions of "local nature"
If p is prime then p has no divisor $1 < d < p^{1/100}$

$$P(x_1, \ldots, x_k) = 0 \quad \underset{\stackrel{?}{\Longleftarrow}}{\Longrightarrow} \quad \forall q \geqslant 1, \quad P(x_1, \ldots, x_k) \equiv O(q)$$

$0 \neq n = x_1^2 + x_2^2$ is solvable if $n > 0$ (Fermat)
$p \mid n$ to odd order $\Longrightarrow p \equiv 1 \pmod 4$

$0 \neq n = x_1^2 + x_2^2 + x_3^2$ is solvable if $n > 0$ (Gauss)

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$n \not\equiv 4^k \pmod{8m+7}$

Counting: $\mathcal{J} \subset \mathbb{N}$
$\sum_{n \leqslant N} 1_{\mathcal{P}}(n)$
$f : \mathbb{N}_{\geqslant 1} \to \mathbb{C}$ evaluation
$\sum_{n \leqslant N} f(n)$ as $N \to \infty$

$f * g(n) = \sum_{ab=n} f(a)g(b)$
$f \boxplus g(n) = \sum_{a+b=n} f(a)g(b)$

Example 1. *Divisor function*

$$d(n) = \sum_{d|n} 1 = 1 * 1(n)$$

Example 2.

$$\Lambda(n) = \begin{cases} \log p, & n = p^\alpha \\ 0 \end{cases} = \mu * \log$$

$\mu(p) = -1$
$\mu(p^\alpha) = 0,\ \alpha \geqslant 2$
$\mu(mn) = \mu(m)\mu(n)$ if $(m, n) = 1$

Form generating series
$F(z) = \sum_n f(n)e(nz) \quad \operatorname{Im} z > 0, z \in \mathbb{H}$
$L(f, s) = \sum_n \frac{f(n)}{n^s} \quad Res \gg 1$
$FG(z) \longleftrightarrow f \boxplus g$
$L(f, s)L(g, s) \longleftrightarrow f * g$

Example 3. $1_\square \quad F(z) = \theta(z)$ *invariance under some* $\Gamma \subset SL_2(\mathbb{Z}) \circlearrowleft \mathbb{H}$
$1_\square \boxplus 1_\square \quad \theta^2(z)$
$d(n) \longleftrightarrow E(z, s)'\Big|_{s=1/z}$

Otherwise: use the circle method
- summing arithmetic functions over short intervals

$$\sum_{n \in [N, N+H[} f(n) \text{ where } H = o(N)$$

- summing arithmetic functions over arithmetic progressions

$$\sum_{\substack{n \leqslant N \\ n \equiv a\ (q)}} f(n)$$

$f \equiv 1_{\mathcal{P}}$: can handle intervals of length $N \exp(-c\sqrt{\log N})$ with $c > 0$
arithmetic progressions of modulus $q \leqslant (\log N)^A$ for all $A \geqslant 0$
GRH can do $H \geqslant N^{1/2}(\log N)^3$, $q \leqslant N^{1/2}(\log N)^{-3}$

Result on average towards

Theorem 1 (Bombieri-Vinogradov)**.** $\forall A \geqslant 0$

$$\sum_{q \leqslant Q} \left| \sum_{\substack{p \equiv a (q) \\ p \leqslant N}} 1 - \frac{1}{\varphi(q)} \sum_{p \leqslant N} 1 \right| \ll_A \frac{N}{(\log N)^A}$$

for $Q \leqslant N^{1/2}/(\log N)B(A)$*.*

To prove this theorem, Zhang went beyond BV.
Fouvry-Iwaniec, Bombieri-Friedlander