MINI-COURSE ON MULTIPLICATIVE FUNCTIONS
PRESENTED BY MAKSYM RADZIWILL

PRIMES $\pi(x) = \sum_{p \le x} 1$

CONSIDER $\sum_{n \le x} \Lambda(n)$ INSTEAD

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \\ 0 & \text{otherwise} \end{cases}$$

PRIME NUMBER THM STATES THAT

$$\sum_{n \le x} \Lambda(n) \sim x$$

$$\Lambda(n) = - \sum_{d \mid n} \mu(d) \log d$$

μ IS A MULTIPLICATIVE FUNCTION $(\mu(mn) = \mu(m)\mu(n) \text{ if } (m,n)=1)$

DEFINED BY $\mu(p^\alpha) = \begin{cases} -1 & \text{if } \alpha = 1 \\ 0 & \text{otherwise} \end{cases}$

PRIME NUMBER THEOREM $\iff \sum_{n \le x} \mu(n) = o(x)$

MORE REFINED QUESTION:

* WHAT IS THE SHORTEST INTERVAL THAT SHOULD CONTAIN A PRIME?
* — / — THAT SHOULD BOTH A NEGATIVE AND POSITIVE VALUE OF μ?

LIOUVILLE FUNCTION: $\lambda(ab) = \lambda(a)\lambda(b)$, $\lambda(p^\alpha) = (-1)^\alpha$

CRAMER MODEL

"A RANDOM INTEGER $X \leq n \leq 2X$ HAS PROB. $\frac{1}{\log X}$

OF BEING PRIME."

" ___ / ___ HAS PROB $\frac{1}{2}$ OF HAVING $\lambda(n) = -1$."

THE PROBABILITY THAT $X \leq x \leq 2X$

$$\mathbb{P}\Big(\Big|\sum_{x \leq n \leq x+h} \Lambda(n)\Big| = 0\Big) \approx \Big(1 - \frac{1}{\log X}\Big)^h$$

$$\mathbb{P}\Big(\Big|\sum_{x \leq n \leq x+h} \lambda(n)\Big| = h+1\Big) = \Big(\frac{1}{2}\Big)^h$$

if $h > C \log^2 X$ (resp $h > C \log X$) THEN BOTH ARE $< \frac{1}{X}$.

SO THE EXPECTED # OF x's $\in [X, 2X]$ FOR WHICH EITHER HOLDS IS < 1.

CONJECTURE: THERE ARE ALWAYS PRIMES IN $[X, X + 10 \log^2 X]$
SIGN CHANGES OF $\lambda(\cdot)$ in $[X, X + 10 \log X]$

* BUT BY THE CENTRAL LIMIT THEOREM, WE EXPECT THAT FOR MOST $x \in [X, 2X]$, $\frac{1}{h} \sum_{x \leq n \leq x+h} \lambda(n) = o(1)$ AS SOON AS $h \to +\infty$.
(=density 1 of $x \in [X, 2X]$)

* ___ / ___ $\frac{1}{h} \sum_{x \leq n \leq x+h} \Lambda(n) \sim 1$ AS SOON AS $\frac{h}{\log X} \to +\infty$.

THM (SELBERG) ASSUME RIEMANN HYPOTHESIS:

$$(*) \quad \frac{1}{h} \sum_{x \le n \le x+h} \Lambda(n) \sim 1$$

FOR ALMOST ALL $x \in [X, 2X]$ AS SOON AS $h > \log^{2+\varepsilon} X$.

REMARK: MAIER PROVED THAT $(*)$ FAILS IF $h < \log^A X$ FOR SOME SEQUENCE OF $X \to +\infty$.

COROLLARY: (MATÖMAKI - R.) FOR ALMOST ALL $x \in [X, 2X]$

$$\frac{1}{h} \sum_{x \le n \le x+h} \lambda(n) = o(1)$$

AS SOON AS $h \to +\infty$ WITH X.

REMARK: CONJECTURALLY THERE ARE ARBITRARILY MANY INTERVALS WITH $\sum_{X \le n \le X+h} \lambda(n) = h+1$ IF $h = o(\log X)$.

THEOREM (MATÖMAKI - R.) LET $f: \mathbb{N} \to [-1, 1]$ BE MULTIPLICATIVE. THEN FOR ALMOST ALL $x \in [X, 2X]$

$$\frac{1}{h} \sum_{x \le n \le x+h} f(n) = \frac{1}{X} \sum_{X \le n \le 2X} f(n) + o(1)$$

AS SOON AS $h \to +\infty$ WITH X.

SOME CONSEQUENCES:

1. EXISTENCE OF x^{ε}- SMOOTH NUMBERS IN $[X, X + c(\varepsilon)\sqrt{x}]$
2. COUNTING SIGN CHANGES OF MULT. FUNCT.
3. (TAO) $\displaystyle\sum_{n \le x} \frac{\lambda(n)\lambda(n+1)}{n} = o(\log x)$

CONJECTURE OF ERDÖS THAT THE LIMIT

$$\lim_{X \to +\infty} \frac{1}{X} \sum_{X \le n \le 2X} f(n) \quad \text{ALWAYS EXISTS}$$

SOLVED BY WIRSING

$$\begin{cases} \text{CASE 1:} & \sum_p \frac{1-f(p)}{p} < +\infty \quad \text{THEN THE } \lim \neq 0 \\ \text{CASE 2:} & \sum_p \frac{1-f(p)}{p} = \infty \quad \text{THEN THE } \lim = 0 \end{cases}$$

<u>THEOREM</u>: ASSUME RIEMANN HYPOTHESIS. THEN

$$\frac{1}{h} \sum_{x \le n \le x+h} \lambda(n) = o(1) \quad \text{FOR ALMOST ALL } x \text{ AS SOON AS } h > X^{\varepsilon}.$$

<u>PROOF</u>: IT'S ENOUGH TO SHOW THAT

$$(*) \qquad \frac{1}{X} \int_X^{2X} \Big| \sum_{x \le n \le x+h} \lambda(n) \Big|^2 dx = o(H^2)$$

FOR $H > X^{\varepsilon}$, BY CHEBYSCHEV INEQUALITY.

(*) IS "EQUIVALENT" TO

$$\int_{\mathbb{R}} \Big| \sum_{x e^{-1/T} \le n \le x e^{1/T}} \lambda(n) \, \mathbb{1}_{X \le n \le 2X} \Big|^2 \frac{dx}{x}, \qquad T = \frac{X}{H}$$

$$\| $$

$$\int_{\mathbb{R}} | f_X(x) |^2 dx$$

$$f_X(x) = \sum_{e^{x - \frac{1}{T}} \le n \le e^{x + \frac{1}{T}}} \lambda(n) \, \mathbb{1}_{X \le n \le 2X}$$

NOTE THAT

$$\hat{f}_X(\xi) = \int_{\mathbb{R}} f_X(x) e^{2\pi i x \xi} dx$$

$$= \sum_{X \le n \le 2X} \lambda(n) \int_{\log n - \frac{1}{T}}^{\log n + \frac{1}{T}} e^{2\pi i x \xi} \, dx$$

$$= \sum_{X \le n \le 2X} \lambda(n) \, n^{2\pi i \xi} \, \frac{e^{2\pi i \xi / T} - e^{-2\pi i \xi / T}}{2\pi i \xi}$$

BY PARSEVAL, WE GET

$$\int_{\mathbb{R}} |f_X(x)|^2 \, dx \asymp \int_{\mathbb{R}} |\hat{f}_X(\xi)| \, d\xi$$

$$= \int_{\mathbb{R}} \Big| \sum_{X \le n \le 2X} \lambda(n) \, n^{2\pi i \xi} \Big|^2 \cdot \underbrace{\left(\frac{\sin 2\pi \xi / T}{2\pi \xi} \right)^2}_{\frac{1}{T^2} \mathbb{1}_{|\xi| \le T}} d\xi$$

WE ESSENTIALLY GOT THAT

$$\frac{1}{X} \int_X^{2X} \Big| \sum_{x \le n \le x+H} \lambda(n) \Big|^2 dx$$

$$\asymp \underbrace{\frac{1}{(X/H)^2} \int_0^{X/H} \Big| \sum_{X \le n \le 2X} \lambda(n) \, n^{2\pi i \xi} \Big|^2 d\xi}_{\text{WANT TO SHOW THIS } o(H^2)}$$

MONTGOMERY - VAUGHAN

$$\int_{-T}^{T} \Big| \sum_{N \le n \le 2N} a(n) \, n^{it} \Big|^2 dt = (T + O(N)) \sum_{N \le n \le 2N} |a(n)|^2$$

APPLYING M-V, WE GET

$$(*) \le \frac{1}{(X/H)^2} \left(\frac{X}{H} + O(X) \right) X = O(H^2)$$

RIEMANN- HYPOTHESIS GIVES

$$\Big| \sum_{X \leq n \leq 2X} \lambda(n)\, n^{2\pi i \xi} \Big| = O_\varepsilon\left(X^{\frac{1}{2}+\varepsilon} \right)$$

THEN

$$(*) \leq \frac{1}{(X/H)^2} \, \frac{X}{H} \left(X^{\frac{1}{2}+\varepsilon} \right)^2 = H X^{2\varepsilon} = o\left(H^2 \right)$$

AS SOON AS $H > X^{3\varepsilon}$.