GEOMETRIC ANALYTIC NUMBER THEORY

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① QUESTIONS IN ANALYTIC NUMBER THEORY OVER $\mathbb{Z}$
(ESP. ARITHMETIC STATISTICS)

② ANALOGOUS QUESTION WITH $\mathbb{Z}$ REPLACED BY $\mathbb{F}_q[t]$
(ANALYTIC NUMBER THEORY OVER FUNCTION FIELDS)

③ ANALOGOUS QUESTION WITH $\mathbb{F}_q[t]$ REPLACED BY $\mathbb{C}[t]$
COUNTING REPLACED BY GEOMETRY

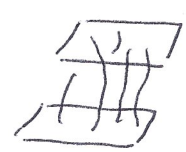
EASY PROBLEM: HOW MANY INTEGERS IN $[X, 2X]$ ARE SQUAREFREE?

$\mathbb{Z}$	$\mathbb{F}_q[t]$	$\mathbb{C}[t]$
interval $[X, 2X]$ # integers in ↑ which are squarefree are $\zeta(2)^{-1}X + O(X^{1/2})$ $\approx \frac{6}{\pi^2} X$	$\lvert f \rvert = q^{\deg f}$ $X = q^n$ interval = f monic of degree n q^n f in this "interval" # of these which are square free is $q^n - q^{n-1}$ $= \left(1 - \frac{1}{q}\right) q^n$ $= \zeta_{\mathbb{F}_q[t]}(2)^{-1} q^n$ $= \lvert \mathrm{Conf}^n(\mathbb{F}_q) \rvert$	interval = f monic in $\mathbb{C}[t]$ of degree n $\mathrm{Conf}^n \mathbb{C}$ = space of squarefree deg n polynomials $= \{t^n + a_{n-1}t^{n-1} + \dots + a_0 : \Delta(a_{n-1}, \dots, a_0) \neq 0\}$ = moduli space of unordered n-tuples of distinct complex #s $z_1, \dots, z_n$ $p \to \{\text{roots of } p\}$ Compute its cohomology

THEOREM (ARNOLD)

$H_0(\mathrm{Conf}^n \mathbb{C}, \mathbb{Q}) = \mathbb{Q}$ ie $\mathrm{Conf}^n \mathbb{C}$ is CONNECTED

$H_1(\mathrm{Conf}^n \mathbb{C}, \mathbb{Q}) = \mathbb{Q}$ $n > 1$



$H_i(\mathrm{Conf}^n \mathbb{C}, \mathbb{Q}) = 0$ $\forall n$

$\mathrm{Conf}^1 \mathbb{C} = \mathbb{C}$; $\mathrm{Conf}^2 \mathbb{C} = S^1$

GROTHENDIECK - LEFSCHETZ TRACE FORMULA:

$$|\mathrm{Conf}^n(\mathbb{F}_q)| = \sum (-1)^i \,\mathrm{Tr}\ \mathrm{Frob} \mid H_{ef}(\mathrm{Conf}^n)$$

$$= \underset{q^n}{\mathrm{Tr}\ \mathrm{Frob}\ H_0} - \underset{q^{n-1}}{\mathrm{Tr}\ \mathrm{Frob} \mid H_1}$$

E, VENKATESH, WESTERLAND

ENTIN, BANK, RODITTY-GERSHON, BORY - SOROKER

RUDNICK, POLLACK, CARDAN ...

A MOMENT OF q

THE SECOND COLUMN IS REALLY NOT ABOUT JUST ONE FIELD BUT A WHOLE FAMILY OF FIELDS

BY WEIL: IF X IS A GEOMETRICALLY IRREDUCIBLE VARIETY OVER $\mathbb{F}_q$ OF DIMENSION n THEN

$$\lim_{q \to \infty} \frac{X(\mathbb{F}_q)}{q^n} = 1$$

e.g. $$\lim_{q \to \infty} \frac{|\mathrm{Conf}^n(\mathbb{F}_q)|}{q^n} = 1$$

$$1 - \frac{1}{q}$$

GEOMETRIC MOLLE/LIMIK

(joint with TRAN, WESTERLAND,

BUILDING IN E, VER , WESTERLAND)

<u>Q</u>: HOW MANY DEGREE d EXTENSIONS OF $\mathbb{Q}$ WITH DISC $\leq X$?

$d=2$: $K/\mathbb{Q} = \mathbb{Q}(\sqrt{m})$ m squarefree

$<X$

CONJECTURE: (LIMIK)?

$N_d(K,X) = \#$ degree d extensions of K WITH disc $< X$

$N_d(K,X) \sim C_{d,K} X$

$d=3$ ✓ DAVENPORT - HEILBRON, $\mathbb{Q}$ DATSKOVSKY-WRIGHT K

$d=4,5$ ✓ BHARGAVA, SHANKAR, WANG

GENERAL d:

(E. VERKESH 2005) $N_d(K,X) < X^{\exp(c\sqrt{\log d})}$

MORE GENERALLY, LET $G \subset S_d$, L/K

$N(K,X,G) = \#$ degree-d extensions, disc $< X$

$\mathrm{Gal}(\hat{L}/\mathbb{Q}) = G$.

CONJECTURE (MOLLEN)?

$$C_{K,G} X^{a(G)} \leq N(K,X,G) \leq C_{K,G} X^{a(G)+\varepsilon}$$

WHEN $G=S_d$, $a(G)=1$

THEOREM (E, Westerland, Tron, 2017)

FOR ALL $q > c_G$

$$N(\mathbb{F}_q(t), X, G) \leq C_{q,G,\varepsilon} X^{a(G)+\varepsilon}$$

IN PARTICULAR,

$$N_d(\mathbb{F}_q(t), X) \leq c X^{1+\varepsilon}$$

COLUMN 1: $N_d(\mathbb{C}, X)$

COLUMN 2: $N_d(\mathbb{F}_q(t), X) =$ set of degree-d branded curves $Y \to \mathbb{P}^1/\mathbb{F}_q$

COLUMN 3: HURWITZ SPACE PARAMETRIZING degree-d $Y \to \mathbb{P}^1/\mathbb{C}$

↑

dimen

WHAT ABOUT

$$\lim_{q \to \infty} q^{-n} N_d(\mathbb{F}_q(t), q^n)$$

This is 1 because Hurwitz space is connected.

VARIANCE OF SHORT SUMS OF ARITHMETIC FUNCTIONS

Q: WHAT IS THE VARIANCE OF

$$\sum_{x \leq n \leq x+H} \mu(n)$$ AS x VARIES OVER $[x, 2x]$?

Keating - Rudnick:

show $\lim_{q \to \infty} H^{-1} \operatorname{Var} = 1$

$H = q^h$

Hest - Matei : more directly showed

this limit existed and can be computed as the dimension of some cohom. group

Rodgers (2016): proved K-R FOR ARBITRARY MULTIPLICATIVE FUNCTIONS

$\downarrow$ Hest - Matei

compute this cohomology group