THE KUZNETSOV FORMULA, KLOOSTERMANIA AND APPLICATIONS

IAN PETROW

$e(z) = e^{2\pi i z}$ $\quad m, n, c \geq 1$ $\quad \sum_{x \in (\mathbb{Z}/c)^*} = \sum^*_{x(c)}$

KLOOSTERMAN SUM: $S(m,n,c) = \sum_{x \in (\mathbb{Z}/c)^\times} e\left(\frac{mx + m x^{-1}}{c}\right)$

$S(m,n,c) = S(-m,-n,c) = \overline{S(m,n,c)} \in \mathbb{R}$

IF $(m,c)=1$, $S(m,n,c) = S(1, mn, c)$

$c = c_1 c_2 \quad (c_1, c_2) = 1$

$S(n\bar{c}_1, n\bar{c}_1, c_2)\, S(m\bar{c}_2, n\bar{c}_2, c_1) = S(m,n,c)$

KLOOSTERMAN (1926) INTRODUCED $S(m,n,c)$ TO STUDY

$r_{abcd}(n) = \#\{x \in \mathbb{Z}^4 \mid n = a x_1^2 + b x_2^2 + c x_3^2 + d x_4^2\}$

LET $Q \in \mathbb{Z}[x_1, \ldots, x_r]$ POS. DET. QUAD. FORM IN $r \geq 4$.

ASSUME r EVEN.

$A = (a_{ij})$ SYMMETRIC MATRIX ASSOCIATED TO Q

$$Q(x) = \frac{1}{2} A[x] = \frac{1}{2} x^{tr} A x = \sum_i a_{ii} x_i^2 + \sum_{i<j}\sum a_{ij} x_i x_j$$

LET $D = (-1)^{r/2} \det A$ DISC. OF Q

LET $N \in \mathbb{N}$ ANY FOR WHICH $NA^{-1} \in M_r(\mathbb{Z})$

LET $\Theta(z) = \sum_{m \in \mathbb{Z}^r} e(zQ(m)) = \sum_{n \geq 0} r(n, Q)\, e(nz)$ $\quad z \in \mathcal{H}$

$\Theta(z)$ IS A MODULAR FORM OF LEVEL N wt $r/2$

CENTRAL CHAR $\left(\frac{D}{\cdot}\right)$

IF $(d,c) = 1$ WRITE $\bar{d} \overset{\text{def}}{=} d^{-1}$ in $(\mathbb{Z}/c)^\times$

$\forall\ \gamma \in \Gamma_0(N)$ $\quad \theta(\gamma z) = \left(\frac{D}{d}\right) (cz+d)^k\, \theta(z)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

STUDY $\quad r(n, Q) = \int_0^1 \theta(z)\, e(-nz)\, dx$

KLOOSTERMAN CIRCLE METHOD

FAREY FRACTIONS OF DEPTH C:

$$\left\{ \frac{a}{c} : (a,c)=1,\ 1 \le c \le C \right\}$$

WRITE $\frac{a'}{c'} < \frac{a}{c} < \frac{a''}{c''}$ ADJACENT FRACTIONS

LET
$$m\left(\frac{a}{c}\right) = \left(\frac{a+a'}{c+c'} , \frac{a+a''}{c+c''} \right]$$
$$= \left(\frac{a}{c} - \frac{1}{c(c+c')} , \frac{a}{c} + \frac{1}{c(c+c'')} \right]$$

EXERCISE: The medians are not Farey fractions of depth C.

EXERCISE:

c', c'' ARE THE UNIQUE NUMBERS SATISFYING:
$$\begin{cases} C - c < c' \le C & ac' \equiv 1\ (c) \\ C - c < c'' \le C & ac'' \equiv -1\ (c) \end{cases}$$

$$a' = \frac{(ac'-1)}{c}$$

$$a'' = \frac{(ac''+1)}{c}$$

LET $f: \mathbb{R} \to \mathbb{C}$ PERIODIC mod 1

$$\int_0^1 f(x)\, dx = \sum_{\substack{0 \le a < c \le C \\ (a,c)=1}} \int_{m\left(\frac{a}{c}\right)} f(x)\, dx$$

$$= \sum\sum \int_{-1/c(c+d')}^{1/c(c+d)} f\left(x - \frac{a}{c}\right) dx \qquad d = c + c' \text{ or } c + c''$$

$$= \sum_{\substack{1 \le c \le C < d \le C+c \\ (d,c)=1}} \int_{-1/cd}^{1/cd} f\left(x - \frac{\bar{d}}{c}\right) dx$$

Then assume $f(-x) = \overline{f(x)}$

$$\Rightarrow \int_0^1 f(x)\,dx = 2 \operatorname{Re} \sum_{c \le C} \int_0^{1/d} \sum^{*}_{\substack{c<d\le C+c \\ cdx<1}} f\left(x - \frac{\bar{d}}{c}\right) dx$$

$$\int_0^1 \theta(z)\, e(-nx)\,dx = 2 \operatorname{Re} \sum_{c \le C} \int_0^{1/cC} \underbrace{\sum_{\substack{C<d\le C+c \\ dxc<1}} \theta\left(z - \frac{\bar{d}}{c}\right) e\left(\frac{\bar{d}n}{c}\right) e(-nx)}_{T(n,c,x)} dx$$

LEMMA: ASSUME $(a,c) = 1$.

$$\theta\left(z - \frac{a}{c}\right) = (\det A)^{-1/2} c^{-r} \left(\frac{i}{z}\right)^{r/2} \sum_{m \in \mathbb{Z}^r} G_m\left(\frac{-\bar{a}}{c}\right) e\left(\frac{\frac{1}{2}A^{-1}[x]}{c^2 z}\right)$$

$$G_m\left(\frac{\bar{a}}{c}\right) \overset{\text{def}}{=} \sum_{h \in (\mathbb{Z}/c)^r} e\left(\frac{\bar{a}}{c}\left(Q(h) + h^{H} m\right)\right)$$

PROOF: POISSON SUMMATION

$$T(n,c,x) = (\det A)^{-1/2} c^{-r} \left(\frac{i}{z}\right)^{r/2} \sum_{m \in \mathbb{Z}^r} \overbrace{\left[\sum_{\substack{C<d\le C+c \\ cdx<1}} G^{*}_m\left(\frac{d}{c}\right) e\left(\frac{n\bar{d}}{c}\right)\right]}^{\substack{\text{Kloosterman sum} \\ \approx \\ T_m(n,c,x)}} e\left(\frac{\frac{1}{2}A^{-1}[m]}{c^2 z}\right)$$

$$T_m(n,c,x) = \sum_{\substack{c<b\le C+c \\ cbx<1}} \sum^*_{\substack{d(c) \\ b\equiv d(c)}} G_m\left(\frac{d}{c}\right) e\left(\frac{n\bar{d}}{c}\right)$$

$$= \frac{1}{c} \sum_{l(c)} \sum_b e\left(\frac{lb}{c}\right) \sum^*_{d(c)} e\left(\frac{-ld}{c}\right) G_m\left(\frac{d}{c}\right) e\left(\frac{n\bar{d}}{c}\right)$$

$$\frac{1}{c} \sum_{C<b\le \min\left(C+c, \frac{1}{cx}\right)} e\left(\frac{lb}{c}\right) \ll \frac{1}{1+|l|}$$

is harmless

<u>LEMMA:</u> IF $(c, 2d \det A) = 1$ THEN

$$G_m\left(\frac{d}{c}\right) = \left(\frac{\det A}{c}\right) \left((-1)^{\frac{c-1}{2}} c\right)^{r/2} e\left(-\frac{\bar{d}}{c} \frac{1}{2} A^{-1}[m]\right)$$

SO UNDER $(c, 2d \det A) = 1$

$$\sum^*_{d(c)} e\left(-\frac{dl}{c}\right) G_m\left(\frac{d}{c}\right) = \left(\frac{\det A}{c}\right) \left((-1)^{\frac{c-1}{2}} c\right)^{r/2} \sum^*_{d(c)} e\left(\frac{-ld + n\bar{d} - \bar{d}\frac{1}{2}A^{-1}[m]}{c}\right)$$

$$= S\left(-l, n - \frac{1}{2}A^{-1}[m], c\right)$$

TO CONCLUDE AN ASYMPTOTIC FORMULA FOR $n \to \infty$ ANY NON TRIVIAL BOUND ON $S(m,n,c)$ SUFFICES.

<u>KLOOSTERMAN:</u> $\displaystyle\sum^*_{m(p)} |S(m,n,p)|^4 < 16p^3$

$$\Rightarrow |S(m,n,p)| < 2p^{3/4}$$

PETERSON (1939)

LET $S_K(q,\chi)$ BE THE $\mathbb{C}$-v.s. of classical holomorphic cusp forms for

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \;\middle|\; c \equiv 0 \ (q) \right\}$$

weight K, central char χ mod q

$$\langle f,g \rangle = \int_{\Gamma_0(q)\backslash\mathcal{H}} f(z)\,\overline{g(z)}\, y^k \frac{dx\,dy}{y^2}$$

LET $\mathcal{B}_K(q,\chi)$ BE AN O.N. BASIS FOR $S_K(q,\chi)$

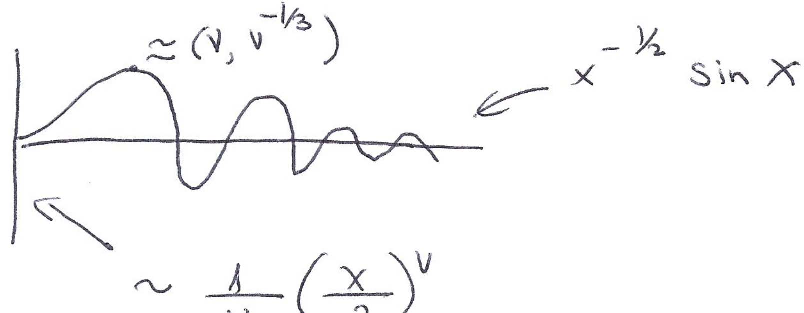
THEOREM: (PETERSSON)

$$\frac{\Gamma(K-1)}{(4\pi m n)^{K-1}} \sum_{f\in \mathcal{B}_K(q,\chi)} \overline{a_f(m)}\, a_f(n)$$

$$= \delta_{m=n} + 2\pi i^{-K} \sum_{\substack{c>0 \\ c\equiv 0(q)}} \frac{S_\chi(m,n,c)}{c} J_{K-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

$$S_\chi(m,n,c) = \sum_{x(c)} \chi(x)\, e\left(\frac{mx+n\bar{x}}{c}\right)$$

$J_\nu(x) =$



BY MULTIPLICATIVITY, IT SUFFICES TO UNDERSTAND $S(m,n,p^{\nu})$. THE CASES $\nu \geq 2$ ARE EASY

q PRIME POWER

LET $f, g \in \mathbb{Z}(x)$

χ DIRICHLET CHAR mod q

ψ ADDITIVE char mod q

STUDY $S(\chi, \psi) = \sum^{*}_{x(q)} \chi(f(x))\, \psi(g(x))$

WRITING $f(x) = \frac{f_1(x)}{f_0(x)}$ $\quad \chi(f(x)) = \chi(f_1(x)\overline{f_0(x)})$

sim. $\psi(g(x))$

LEMMA: LET $q = p^{2\alpha}$, $\alpha \geq 1$. THEN

$$S(\chi, \psi) = p^{\alpha} \sum^{*}_{\substack{y(p^{\alpha}) \\ h(y) \equiv 0\ (p^{\alpha})}} \chi(f(y))\, \psi(g(y))$$

LET $\psi(x) = e\left(\frac{ax}{p^{\alpha}}\right)$, $\chi(1+zp^{\alpha}) = e\left(\frac{bz}{p^{\alpha}}\right)$

$$h(y) = a g'(y) + b \frac{f'}{f}(y).$$

PROOF: WRITE $x = y + zp^{\alpha}$

$$S(x, \psi) = \sum^{*}_{y(p^{\alpha})} \sum_{z(p^{\alpha})} \chi(f(x))\, \psi(g(x))$$

$$f(x) \equiv f(y) + f'(y) z p^{\alpha} \quad (p^{2\alpha})$$

$$\chi(f(x)) = \chi(f(y))\, \chi\left(1 + \frac{f'}{f}(y) z p^{\alpha}\right) = \chi(f(y))\, e\left(\frac{bzf'}{p^{\alpha}f}\right)$$

$$\psi(g(x)) = \psi(g(y))\,\psi(g'(y)\,z p^{\alpha})$$
$$= \psi(g(y))\, e\left(\frac{a z\, g'(y)}{p^{\alpha}}\right)$$

SIMILAR LEMMA IF $p^{2\alpha+1}$, $\alpha \geq 1$.

WEIL BOUND: $|S(m,n,c)| \leq (m,n,c)^{1/2} c^{1/2} d(c)$

<u>THEOREM:</u> LET Q BE A POS-DET QUADRATIC FORM IN $r \geq 4$ VAR.

$$r(n,Q) = \frac{(2\pi)^{k}\, n^{k-1}}{\Gamma(k)(\det A)^{1/2}}\, \mathfrak{S}(n,Q) + O_{\varepsilon,Q}\left(n^{k/2 - 1/4 + \varepsilon}\right)$$

WHERE $$\mathfrak{S}(n,Q) = \sum_{c \geq 1} c^{-r} \sum_{d(c)}^{*} \sum_{n(c)} e\left(\frac{d}{c}\left(Q(n) - n\right)\right)$$

$(\mathbb{Z}/c)^r$