ℓ-ADIC TRACE FUNCTIONS IN ANALYTIC NUMBER THEORY

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ℓ-ADIC COHOMOLOGY IN ANALYTIC NUMBER THEORY

ANOTHER SOURCE FOR KLOOSTERMAN SUMS

$$d(n) = \sum_{d|n} 1 \qquad q \text{ PRIME} \qquad (a,q)=1$$

$$A_d(X,q,a) = \sum_{\substack{n\equiv a(q)\\ n\leq X}} d(n) - \frac{1}{q}\sum_{n\leq X} d(n) = O\left(\frac{X}{q}\frac{1}{(\log X)}\right)$$

<u>THEOREM</u> (SELBERG): $\forall A \geq 0$ AND $q \leq X^{\frac{2}{3}-\varepsilon}$

$$A_d(X,q,a) \ll \frac{X}{q}\frac{1}{(\log X)^A}$$

<u>PROOF</u>: MODEL SUM

$$\sum_{m,n} \delta_{mn\equiv a(q)} V\left(\frac{m}{M}\right) V\left(\frac{n}{N}\right) \qquad MN = X, \quad V \in \mathcal{C}_c^\infty(]1,2[)$$

POISSON: $K = \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ $\qquad e_q(x) = \exp\left(\frac{2\pi i x}{q}\right)$

$$\hat{K}(Y) = \frac{1}{q^{1/2}} \sum_{x(q)} K(x)\, e_q(xy) \qquad \text{the FT of } K$$

$$\sum_{n\in\mathbb{Z}} K(n)\, V\left(\frac{n}{N}\right) = \frac{N}{q^{1/2}} \sum_{n'} \hat{K}(n')\, \hat{V}\left(\frac{N}{q}n'\right)$$

WE APPLY POISSON ON m,n TO THE SUM

$$\sum_{m,n} K(mn)\, V\left(\frac{m}{M}\right) V\left(\frac{n}{N}\right)$$

$$= \frac{\hat{K}(0)}{q^{1/2}} \sum_{m,n} V\left(\frac{m}{M}\right) V\left(\frac{n}{N}\right) + \frac{MN}{q} \sum_{m,n} \check{K}(m,n)\, \hat{V}\left(\frac{M}{q}m\right) \hat{V}\left(\frac{N}{q}n\right)$$

WHERE $\check{K}(n) = \begin{cases} \dfrac{1}{q^{1/2}} \displaystyle\sum_{xy=n} \hat{K}(x)\, e_q(y) & \text{if } (n,q)=1 \\[2ex] K(0) - \dfrac{\hat{K}(0)}{q^{1/2}} & \text{if } q \mid n \end{cases}$

$$K(n) = \delta_{n \equiv a(q)} \qquad \check{K}(n) = \begin{cases} \dfrac{1}{q} \displaystyle\sum_{xy = n(q)} e_q(qx+y) & (n,q)=1 \\[2ex] -\dfrac{1}{q} \end{cases}$$

$$Kl_2(n,q) = \frac{1}{\sqrt{q}} \sum_{\substack{x,y\,(q) \\ xy \equiv n(q)}} e_q(x+y)$$

$$= \frac{1}{\sqrt{q}}\, S(1,n,q)$$

WEIL BOUND $\quad |Kl_2(n,q)| \leq 2$

$$\sum_{m,n} \delta_{mn \equiv a(q)}\, V\left(\frac{m}{M}\right) V\left(\frac{n}{N}\right)$$

$$= \left(\frac{1}{q} + \frac{1}{q^2}\right) \sum_{m,n} V\left(\frac{m}{M}\right) V\left(\frac{n}{N}\right) + O(q^{1/2}) = o\left(\frac{X}{q}\right) \text{ if } q = o(X^{2/3})$$

$$+ \frac{MN}{q^{3/2}} \sum_{(mn,q)=1} \overbrace{Kl_2(amn,q)\, \hat{V}\left(\frac{Mm}{q}\right) \hat{V}\left(\frac{Nn}{q}\right)}$$

RMK: FOUVRY HAS PROVEN THAT ONE CAN HAVE THE BOUND ON AVERAGE OVER q FOR $X^{\frac{2}{3}+\varepsilon} \leq q \leq X^{1-\varepsilon}$

FOUVRY IWANIEC BRIDGED THE $\frac{2}{3}$ GAP ON AVERAGE OVER WELL FACTORABLE MODULI.

$$d_3(n) = \sum_{abc=n} 1$$

$$\Delta_{d_3}(x, q, a)$$

THEOREM (FRIEDLANDER - IWANIEC) $\exists \delta > 0 \ \forall A \geq 0$

$$\Delta_{d_3}(x, q, a) \leq \frac{X}{q} \cdot \frac{1}{(\log x)^A}$$

FOR $q \leq X^{\frac{1}{2}+\delta}$

$\delta < \frac{1}{46}$

THE BULK OF THE PROOF IS TO PROVE THE FOLLOWING

L, M, N $\qquad LMN = q$

$$\sum_{\substack{l,m,n \\ (lmn,q)=1}} Kl_3(alm n, q)\, \hat{V}\left(\frac{l}{L}\right) \hat{V}\left(\frac{m}{M}\right) \hat{V}\left(\frac{n}{N}\right)$$

$$\stackrel{?}{=} O(q^{1-\eta}) \quad \text{FOR SOME } \eta > 0$$

$$Kl_3(n, q) = \frac{1}{q} \sum_{xyz \equiv n\,(q)} e_q(x+y+z)$$

$$|Kl_3| \leq 3$$

FURTHER EXAMPLES OF TRACE FUNCTIONS

$a(q)$ $\qquad x \to e_q(ax)$

$\chi(q)$ DIRICHLET CHAR $\qquad x \to \chi(x)$

l- ADIC TRACE FUNCTIONS ARE CONSTRUCTED OUT l- ADIC SHEAVES ON $\mathbb{P}^1_{\mathbb{F}_q}$

$K = \mathbb{F}_q(x)$ $\qquad K^{sep}$

$$G^{arith} = Gal(K^{sep}/K) \supset G^{geom} = Gal(K^{sep}/\bar{\mathbb{F}}_q . K)$$

GIVEN $U \subset \mathbb{P}^1_{\mathbb{F}_q}$ NONEMPTY OPEN SET

$\ell \nmid q$ ON ℓ-ADIC SHEAF $\mathcal{F}$ IS A FINITE DIM REP

$\rho_{\mathcal{F}} : G^a \to GL(V)$ $\quad V = \overline{\mathbb{Q}}_\ell$ - FINITE DIM

$\forall p$

WHICH IS UNRAMIFIED AT EVERY POINT $x \in U(\overline{\mathbb{F}}_q)$, $x \in U(\mathbb{F}_{q^n})$

$$1 \to I_x \to D_x \to Gal(\overline{\mathbb{F}}_q / \mathbb{F}_{q,n}) \to 1$$

CAN USE THE VOCABULARY OF REP THEORY IRRED SHEAF, ISOTYPIC SHEAF GEOMETRICALLY ''

GIVEN $\mathcal{F}$ A SHEAF (lisse on some U)

THE TRACE FUNCTION OF $\mathcal{F}$:

$$x \in U(\mathbb{F}_q) \to K_{\mathcal{F}}(x) = tr(Frob_x \mid V) \in \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$$

$\mathbb{F}_q = \mathbb{A}^1(\mathbb{F}_q)$

WE EXTEND $K_{\mathcal{F}}$ TO $\mathbb{A}^1(\mathbb{F}_q)$ $\Big\langle$ by 0 / by $tr(Frob_x \mid V^{I_x})$

- $\mathcal{F}$ $\quad \rho_{\mathcal{F}} \to \rho_{\mathcal{F}}^*$ $\quad \mathcal{F}^*$ $\quad K_{\mathcal{F}^*}(x) = tr(Fr_x^{-1} \mid V)$

- $\mathcal{F}, \mathcal{G}$ $\quad K_{\mathcal{F} \otimes \mathcal{G}}(x) = K_{\mathcal{F}}(x) \, K_{\mathcal{G}}(x)$

- $f : \mathbb{P}^1 \to \mathbb{P}^1$ $\quad \mathbb{F}_q(f(x)) \hookrightarrow \mathbb{F}_q(x)$

$$G_k(K^{sep}/K) \hookrightarrow Gal(K^{sep} / \mathbb{F}_q(f(x)))$$

$f^* F$ $\quad K_{f^* F}(x) = K_F(f(x))$

$- \rho_{\mathcal{F}} : G^{a} \longrightarrow GL(V) \xrightarrow[\rho']{} GL(W)$

$$\rho'(\mathcal{F}) = \rho'(\rho_{\mathcal{F}})$$

$$K_{\rho'(\mathcal{F})}(x) = tr(\rho(\rho_{\mathcal{F}}(Frob_x)) | W)$$