A glimpse at arithmetic quantum chaos

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Classical mechanics vs. quantum mechanics

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	classical mechanics	**quantum mechanics**
phase space	SM $T_0M/\{h : h > 0\}$	
moving particle	$f : \mathbb{R} \to SM$ smooth	
bounded observable	$a : SM \to \mathbb{R}$ smooth	
unbounded observable	$a : T_0M \to \mathbb{R}$ smooth	
energy	$H : T_0M \to \mathbb{R}$ $(x, \xi) \mapsto \|\xi\|$	
time evolution	$G^t : SM \to SM$ geodesic flow	

Classical mechanics vs. quantum mechanics

Consider a compact orientable Riemannian manifold M, and consider a particle moving freely on M with unit speed.

	classical mechanics	**quantum mechanics**
phase space	SM $T_0M/\{h : h > 0\}$	$L^2(M)$ Hilbert space
moving particle	$f : \mathbb{R} \to SM$ smooth	$\psi : \mathbb{R} \to L^2(M)$ $\|\psi\| = 1$
bounded observable	$a : SM \to \mathbb{R}$ smooth	$\mathrm{Op}(a) : L^2(M) \to L^2(M)$ self-adjoint & bounded
unbounded observable	$a : T_0M \to \mathbb{R}$ smooth	$\mathrm{Op}(a) : L^2(M) \to L^2(M)$ self-adjoint
energy	$H : T_0M \to \mathbb{R}$ $(x, \xi) \mapsto \|\xi\|$	$\hat{H} : L^2(M) \to L^2(M)$ $\psi \mapsto \sqrt{\Delta}\psi$
time evolution	$G^t : SM \to SM$ geodesic flow	$U_t : L^2(M) \to L^2(M)$ $U_t = e^{-it\sqrt{\Delta}}$

Wigner measure and quantum limits

Orthonormal Laplace eigenbasis of $L^2(M)$

$$\Delta\phi_j = \lambda_j\phi_j, \qquad 0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$$

Solutions of the Schrödinger equation

$$\psi(t) = U_t(\psi(0)) = \sum_{j=0}^{\infty} c_j e^{-it\sqrt{\lambda_j}}\phi_j, \qquad (c_j)_{j=0}^{\infty} \in \ell^2(\mathbb{N})$$

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Wigner measures on SM

- $\langle \mathrm{Op}(a)\phi_j, \phi_j\rangle = \int_{SM} a\, d\omega_j$ *for* $a \in C^\infty(SM)$
- $d\omega_j$ *restricts to* $d\mu_j := |\phi_j|^2 d\mu$ *on* M

Question

What are the possible weak limits of the Wigner measures* $d\omega_j$*?*

Weyl's law

Local Weyl law

$$N(\lambda, a) := \sum_{\lambda_j \leqslant \lambda} \int_{SM} a \, d\omega_j \sim \lambda^{\dim(M)/2} \frac{B_d \operatorname{vol}(M)}{(2\pi)^d} \int_{SM} a \, d\omega$$

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Corollary

The Wigner measures $d\omega_j$ tend to the normalized Liouville measure $d\omega$ in the Cesàro sense:

$$\frac{1}{N(\lambda, 1)} \sum_{\lambda_j \leqslant \lambda} \int_{SM} a \, d\omega_j = \frac{N(\lambda, a)}{N(\lambda, 1)} \to \int_{SM} a \, d\omega$$

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Corollary

If the Wigner measures $d\omega_j$ converge along a subsequence of λ_j's of density 1, then the limit is the normalized Liouville measure $d\omega$.

Egorov's theorem

Theorem (Egorov 1969)

For a given $a \in C^\infty(SM)$ *and* $t \in \mathbb{R}$, *let us write*

$$\mathrm{Op}(a \circ G^t) = U_{-t}\,\mathrm{Op}(a)U_t + K(a,t).$$

Then $\sqrt{\Delta}K(a,t)$ *is a bounded operator from* $L^2(M)$ *to* $L^2(M)$.

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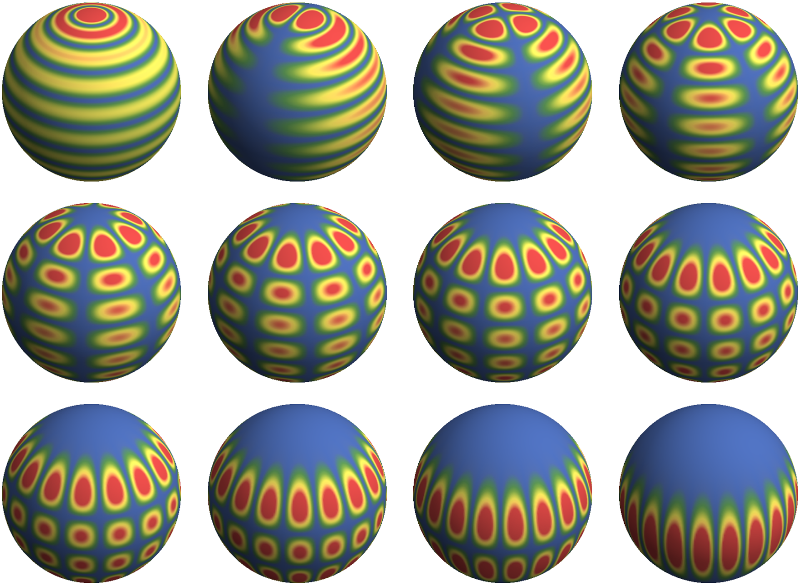
Corollary

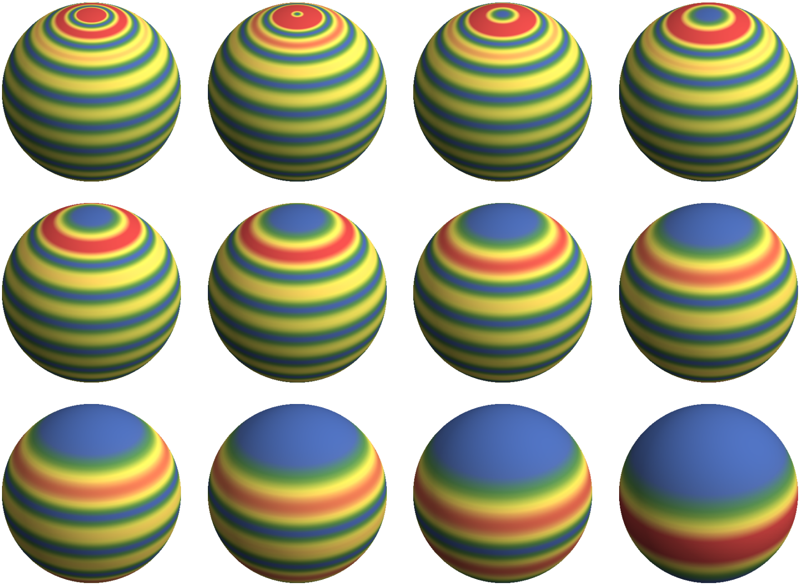
Quantum limits on SM are invariant under the geodesic flow.

Standard real spherical harmonics of degree 11



Mathematica® code credit: Vitaliy Kaurov

Standard complex spherical harmonics of degree 11



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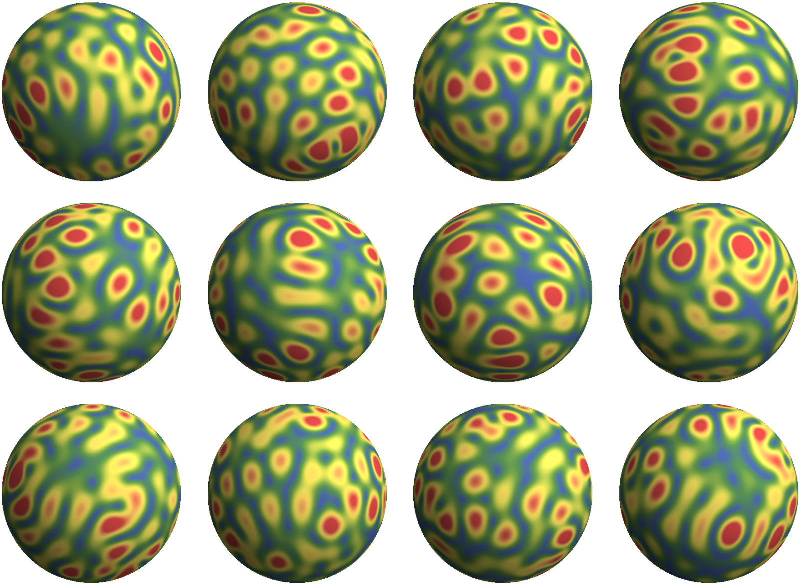

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Quantum limits on the sphere (1 of 2)

Theorem (Uribe 1985, Zelditch 1990)

Let $\{\phi_j\}$ be the standard basis of $L^2(M)$. Consider the orbits in SM of the joint action of the geodesic flow and the rotation group around the vertical axis. The uniform measure on each orbit is a quantum limit on SM, and these are all the quantum limits on SM.

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Theorem (Jakobson–Zelditch 1999)

Every probability measure on SM invariant under the geodesic flow is a quantum limit for some Δ-eigenbasis $\{\phi_j\}$ of $L^2(M)$.

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Every probability measure on SM invariant under the geodesic flow is a quantum limit for some Δ-eigenbasis $\{\phi_j\}$ of $L^2(M)$.

Theorem (Zelditch 1992, VanderKam 1997)

For almost all Δ-eigenbases $\{\phi_j\}$ of $L^2(M)$, the normalized Liouville measure $d\omega$ is the only quantum limit on SM.

Quantum limits on the sphere (2 of 2)

Averaging operators on the sphere

For a finite set of rotations $R \subset \mathrm{SO}_3(\mathbb{R})$, consider the operator

$$T_R\phi(m) := \frac{1}{2|R|}\sum_{r\in R}\left(\phi(rm)+\phi(r^{-1}m)\right), \qquad \phi \in L^2(M).$$

Let $\{\phi_j\}$ be an orthonormal (Δ, T_R)-eigenbasis of $L^2(M)$.

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For a finite set of rotations $R \subset \mathrm{SO}_3(\mathbb{R})$*, consider the operator*

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Let $\{\phi_j\}$ *be an orthonormal* (Δ, T_R)*-eigenbasis of* $L^2(M)$*.*

Theorem (Jakobson–Zelditch 1999)

If R *satisfies a mild technical assumption, then no quantum limit on* SM *is supported on a closed geodesic.*

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Theorem (Jakobson–Zelditch 1999)

If R satisfies a mild technical assumption, then no quantum limit on SM is supported on a closed geodesic.

Theorem (Brooks–Masson-Lindenstrauss 2016)

If R generates a free subgroup of $\mathrm{SO}_3(\mathbb{R})$, then on M the projected measures $d\mu_j = |\phi_j|^2 d\mu$ converge to $d\mu$ along a subsequence of λ_j's of density 1.

Arithmetic quantum limits on the sphere (1 of 2)

Let us identify $\mathcal{S}^2$ with $\{x\mathbf{i} + y\mathbf{j} + z\mathbf{k} : \; x^2 + y^2 + z^2 = 1\}$, then each nonzero quaternion γ acts on $\mathcal{S}^2$ via $m \mapsto \gamma m \overline{\gamma}/(\gamma\overline{\gamma})$. Let

$$\mathcal{O} := \left\{ \frac{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}}{2} : \; a, b, c, d \in \mathbb{Z}, \; a \equiv b \equiv c \equiv d \bmod 2 \right\}$$

be the ring of Hurwitz quaternions.

The 24 units of this ring act on $\mathcal{S}^2$ by the group $\mathcal{O}^\times/\{\pm 1\} \cong A_4$.

A fundamental domain is the spherical quadrangle $T_1 \cup T_2$ in the picture.

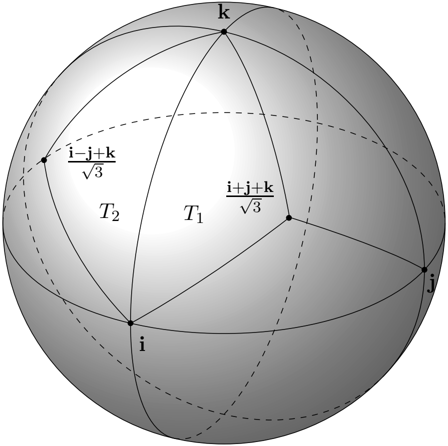

Image credit: Michael Magee

Arithmetic quantum limits on the sphere (2 of 2)

Hecke operators on the sphere

Let $M := \mathcal{O}^\times \backslash \mathcal{S}^2$*. For a prime* $p > 2$*, consider the Hecke operator*

$$T_p \phi(m) := \frac{1}{\sqrt{p}} \sum_{\substack{\gamma \in \mathcal{O}^\times \backslash \mathcal{O} \\ \gamma\bar{\gamma} = p}} \phi(\gamma . m), \qquad \phi \in L^2(M).$$

Let $\{\phi_j\}$ *be an orthonormal Hecke eigenbasis of* $L^2(M)$*.*

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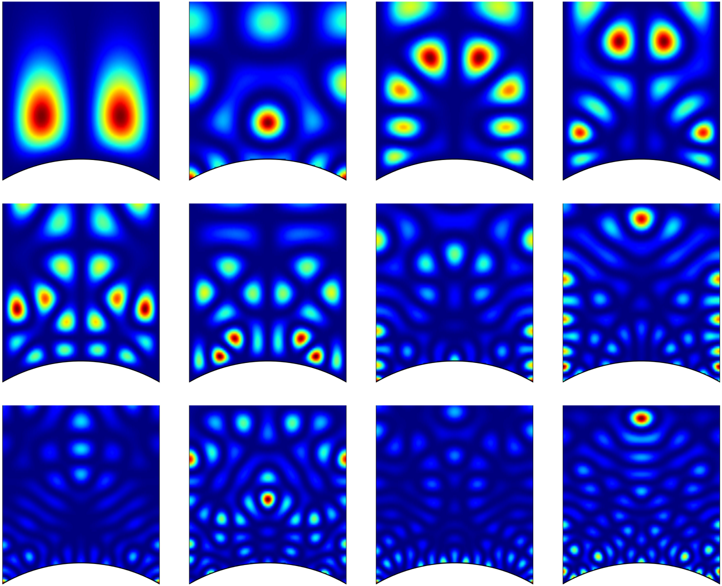
Theorem (Böcherer–Sarnak–Schulze-Pillot 2003)

For $j \geqslant k \geqslant 1$ *we have*

$$\left| \int_M \phi_k \, d\mu_j \right|^2 \ll_\varepsilon (\lambda_j \lambda_k)^{-1/2+\varepsilon} L\left(\frac{1}{2}, f_j \otimes \tilde{f}_j \otimes f_k\right),$$

where f_j *and* f_k *are the holomorphic cuspidal newforms associated to* ϕ_j *and* ϕ_k *by the Eichler/Jacquet–Langlands correspondence. In particular, GRH (or subconvexity) implies that* $d\mu_j \xrightarrow{*} d\mu$*.*

Maass forms on the modular surface



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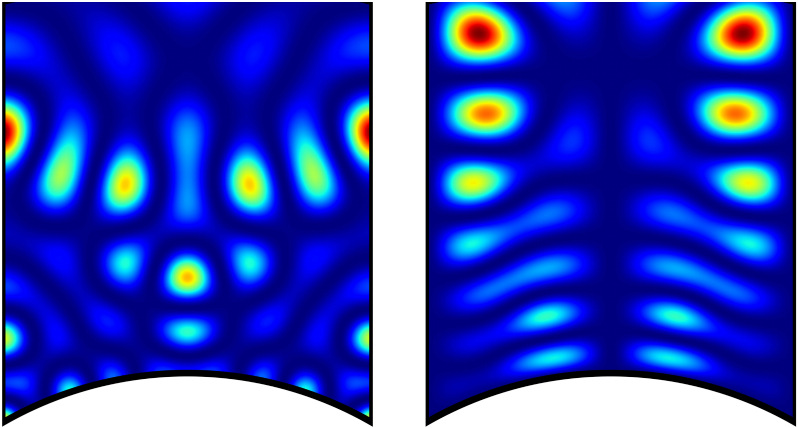
Maass forms with $\lambda \approx 10^3$ on the modular surface



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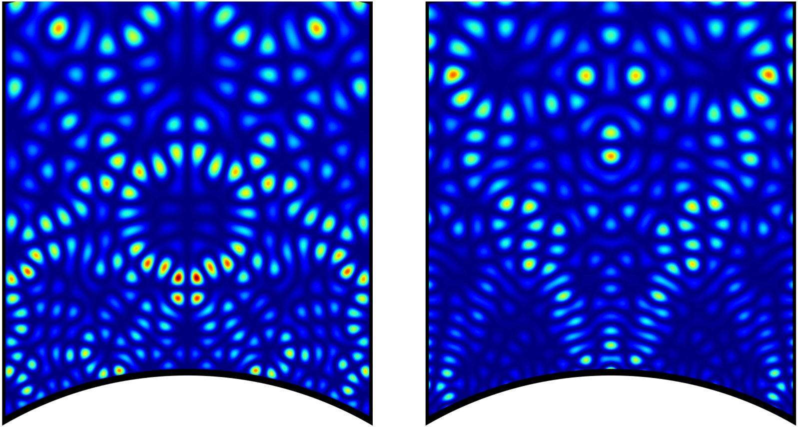
Maass forms with $\lambda \approx 10^4$ on the modular surface



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Maass forms with $\lambda \approx 10^5$ on the modular surface

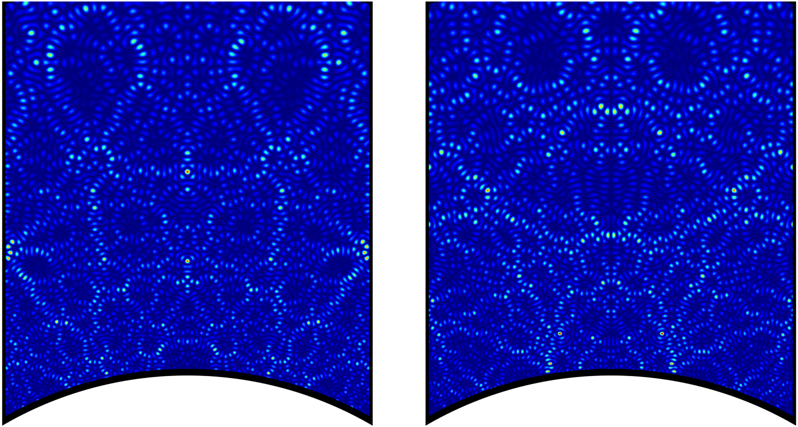

Image credit: Fredrik Strömberg

Quantum ergodicity on the modular surface (1 of 2)

Theorem (Hopf 1936)

Let $M := \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^2$ be the modular surface. The geodesic flow on SM is ergodic.

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Proof (sketch).

By Iwasawa, SM can be identified with $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$, and then the geodesic flow G^t acts by right multiplication by $\begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$.

Assume that $f \in L^2(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}))$ is fixed by this action. Then, for any fixed $b \in \mathbb{R}$ and for $a > 0$ tending to infinity,

$$\begin{aligned} \left\| \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} f - f \right\| &= \left\| \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f - \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f \right\| \\ &= \left\| \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f - f \right\| \\ &= \left\| \begin{pmatrix} 1 & a^{-1}b \\ & 1 \end{pmatrix} f - f \right\| \to \|f - f\| = 0. \end{aligned}$$

Hence any upper triangular matrix in $\mathrm{SL}_2(\mathbb{R})$ fixes f. Similarly, any lower triangular matrix in $\mathrm{SL}_2(\mathbb{R})$ fixes f. In the end, the entire group $\mathrm{SL}_2(\mathbb{R})$ fixes f, and so f is constant almost everywhere. □

Quantum ergodicity on the modular surface (2 of 2)

Theorem (Shnirelman 1974, Colin de Verdière 1985, Zelditch 1987)

Assume that the geodesic flow on SM is ergodic, and let $\{\phi_j\}$ *be any orthonormal Laplace eigenbasis of* $L^2(M)$. *Then* $d\omega_j \xrightarrow{*} d\omega$ *along a subsequence of* λ_j*'s of density* 1.

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Proof (sketch).

Assume that $a \in C^\infty(SM)$ has space average $\int_{SM} a\, d\omega = 0$.
Consider also a fixed time average $a^T := \frac{1}{T}\int_0^T a \circ G^t\, dt$.
By Egorov, Cauchy–Schwarz, Weyl, and Birkhoff, we have

$$\frac{1}{N(\lambda,1)} \sum_{\lambda_j \leqslant \lambda} \left| \int_{SM} a\, d\omega_j \right|^2 = \frac{1}{N(\lambda,1)} \sum_{\lambda_j \leqslant \lambda} \left| \int_{SM} a^T\, d\omega_j \right|^2 + o(1)$$

$$\leqslant \frac{1}{N(\lambda,1)} \sum_{\lambda_j \leqslant \lambda} \int_{SM} |a^T|^2\, d\omega_j + o(1) = \int_{SM} |a^T|^2\, d\omega + o(1) < \varepsilon,$$

for $T = T_0(\varepsilon)$ and $\lambda > \lambda_0(\varepsilon)$. Hence the left hand side is $o(1)$. □

The quantum unique ergodicity conjecture

Assume that M has negative sectional curvature, and let $\{\phi_j\}$ be any orthonormal Laplace eigenbasis of $L^2(M)$.

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Theorem (Anantharaman 2008)

Quantum limits on SM have positive entropy for the geodesic flow.

Theorem (Hassell–Hillairet 2010)

In the above conjecture, it is not enough to assume that M has nonpositive sectional curvature with ergodic geodesic flow on SM.

Arithmetic QUE on the modular surface (1 of 2)

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Let $M := \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}^2$. *For a prime* p, *consider the Hecke operator*

$$T_p\phi(m) := \frac{1}{\sqrt{p}} \sum_{\substack{\gamma\in \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{M}_2(\mathbb{Z}) \\ \det\gamma = p}} \phi(\gamma.m), \qquad \phi \in L^2(M).$$

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Let $\{\phi_j\}$ *be an orthonormal Hecke eigenbasis of* $L^2(M)$.

Theorem (Watson 2002)

For $j \geqslant k \geqslant 1$ *we have*

$$\left|\int_M \phi_k \, d\mu_j\right|^2 \ll_\varepsilon (\lambda_j\lambda_k)^{-1/2+\varepsilon} L\left(\frac{1}{2}, \phi_j \otimes \tilde{\phi}_j \otimes \phi_k\right).$$

In particular, GRH (or subconvexity) implies that $d\mu_j \xrightarrow{*} d\mu$.

Arithmetic QUE on the modular surface (2 of 2)

Theorem (Lindenstrauss 2006, Soundararajan 2010)

The arithmetic quantum unique ergodicity conjecture is true on the modular surface (or on any arithmetic hyperbolic surface).

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The arithmetic quantum unique ergodicity conjecture is true on the modular surface (or on any arithmetic hyperbolic surface).

Theorem (Brooks–Lindenstrauss 2016)

In the above theorem, a single Hecke operator T_p suffices. More precisely, if $\{\phi_j\}$ is a (Δ, T_p)-eigenbasis of $L^2(M)$, then on M the projected measures $d\mu_j = |\phi_j|^2 d\mu$ converge to $d\mu$.