Quadratic twists of elliptic curves with 3-torsion

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(Joint with M. Bhargava, Z. Klagsbrun, and A. Shnidman)

¹Partially supported by NSF grant DMS-1601398

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Theorem (Mazur)

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- $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$ for $N \in \{2, 4, 6, 8\}$.

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• What is Prob.(r(E) = m) for given $m \ge 0$?

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- What is $\operatorname{Prob}(r(E) = m)$ for given $m \ge 0$?
- What is Avg.(r(E))?

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Corollary (Parity conjecture)

Assuming BSD, for any E/\mathbb{Q} ,

$$(-1)^{r(E)}=w(E).$$

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w(E)=1 for 50% of E/\mathbb{Q} and w(E)=-1 for 50% of E/\mathbb{Q} .

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Theorem (Bhargava-Shankar, Bhargava-Skinner-Zhang)

At least 20% of E/\mathbb{Q} have rank 0, and at least 20% have rank 1.

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Question (Weak Goldfeld)

Do $r(E_d) = 0$ and $r(E_d) = 1$ hold for a positive proportion of d?

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Theorem (Smith)

 $r(E_d) = 1$ for at least 27.9% of d unconditionally.

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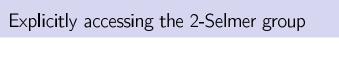
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Idea (Heath-Brown)

Study the distribution of $s_2(E_d)$.



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(Secretly, what's going on is that

$$\begin{array}{rcl} \operatorname{Sel}_2(E_d/\mathbb{Q}) & \subseteq & H^1(\mathbb{Q}, E_d[2]) \\ & \simeq & H^1(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \\ & \leftrightarrow & \text{biquadratic fields.} \end{array}$$

But let's agree to mostly ignore this picture!)

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Theorem (Kane, Swinnerton-Dyer)

If $E(\mathbb{Q})[2] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then:

- the average rank of E_d/\mathbb{Q} is at most 3/2,
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Question

How common are curves with $E(\mathbb{Q})[2] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$?

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There are $\sim \frac{4}{\zeta(10)} X^{5/6}$ non-isomorphic E/\mathbb{Q} with $Ht(E) \leq X$.

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There are $\asymp X^{1/3}$ non-isomorphic curves with $Ht(E) \le X$ and $E(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

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Question

Are there bigger families for which we can prove weak Goldfeld?

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Remark

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Remark

It follows from Harron and Snowden that there are $\approx X^{1/2}$ such curves.

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Explicit Examples

Corollary

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Corollary

Let $E: y^2 + y = x^3 + x^2 + x$. Then E has a 3-isogeny, and

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Remark

The above corollary also holds for $E: y^2 = x^3 + 208$ and $E: y^2 + y = x^3 + x^2 - 3x + 1$, among others.

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E has a 3-isogeny iff there exists d such that $E_d(\mathbb{Q})[3] \simeq \mathbb{Z}/3\mathbb{Z}$.

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In fact, let E': $y^2 + y = x^3 + x^2 - 9x - 15$, and define

$$\phi((x,y)) = \left(\frac{x^3 + 2x + 1}{x^2}, \frac{x^3 - 2xy - x - 2y - 1}{x^3}\right).$$

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If $\phi \colon E \to E'$ is an isogeny, then r(E) = r(E'). We will use the ϕ -Selmer group to access the rank.

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Let $\phi \colon E \to E'$ be a 3-isogeny. The average size of $\mathrm{Sel}_{\phi}(E_d/\mathbb{Q})$ is exactly

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Better: Elements of $Sel_{\phi}(E_d/\mathbb{Q})$ correspond to binary cubic forms.

Theorem (Delone-Fadeev)

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Proof.

Use Delone-Fadeev and geometry of numbers.

Binary cubic forms and ϕ -Selmer elements

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Idea

Use geometry of numbers + theorem to count ϕ -Selmer elements.

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Question

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Question

Are we just inefficient, or is there something weird with ϕ -Selmer?

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Corollary

For E/\mathbb{Q} with a 3-isogeny, $\operatorname{Avg.}|\operatorname{Sel}_{\phi}(E/\mathbb{Q})| \gg (\log X)^{2/3}$.

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