MINI-COURSE ON MULTIPLICATIVE FUNCTIONS II

PRESENTED BY MAKSYM RADZIWILL

OPEN PROBLEM: ALMOST ALL INTERVALS $[x, x+x^{\varepsilon}]$ CONTAIN AT LEAST ONE PRIME.

THEOREM (MOTOHASHI, '70'S) FOR ALMOST ALL $X \leq x \leq 2X$ THERE IS A PRODUCT OF EXACTLY TWO PRIMES IN $[x, x+x^{\varepsilon}]$

LET'S FOCUS ON $n = pq$ WITH $p \approx A$, $q \approx B$

$AB = X$

THE TOTAL NUMBER OF SUCH $pq \in [X, 2X]$ IS $\approx \pi(A)\pi(B)$

$$\geq \frac{X}{\log^2 X}$$

WE WANT TO SHOW THAT

$$(*) \quad \frac{1}{X}\int_X^{2X} \Bigg| \frac{1}{H} \sum_{\substack{x \leq pq \leq x+H \\ p \approx A \\ q \approx B}} 1 \; - \; \frac{1}{X} \sum_{\substack{X \leq pq \leq 2X \\ p \approx A \\ q \approx B}} 1 \Bigg|^2 dx = o\left(\frac{1}{\log^{A} X}\right)$$

$H = X^{\varepsilon}$, $A = 100$

PARSEVAL:

$$(**) \quad (*) \leq \frac{1}{X^2} \int_{(\log X)^{10}}^{X/H} \Bigg| \sum_{p \approx A} p^{it} \cdot \sum_{q \approx B} q^{it} \Bigg|^2 dt$$

BY THE PRIME NUMBER THEOREM

$$\sum_{p \approx A} p^{it} = o\left(A \cdot \exp(-(\log A)^{\varepsilon}\right)$$

PROVIDED THAT $X > A > \exp\left((\log X)^{2/3+\varepsilon}\right)$ FOR $\underset{\substack{\wedge \\ X^2}}{|t|} > \log^{10} X$.

$$(**)\quad \frac{A^2\exp(-(\log A)^{\varepsilon})}{X^2}\int_{(\log X)^{10}}^{X/H}\Big|\sum_{q\approx B} q^{it}\Big|^2\,dt$$

APPLY MONTGOMERY – VAUGHAN

$$\int_{-T}^{T}\Big|\sum_{N\le n\le 2N} a(n)\,n^{it}\Big|^2 dt \le (T+O(N))\sum_{N\le n\le 2N}|a(n)|^2$$

$$\le \frac{A^2\exp(-(\log A)^{\varepsilon})}{X^2}\left(\frac{X}{H}+B\right)B$$

WE GET $(*)\le \underbrace{\underbrace{\frac{A^2B}{XH}\exp(-(\log A)^{\varepsilon})}_{\frac{A}{H}\exp(-(\log A)^{\varepsilon})}}_{\text{THIS FORCES } A\le H} + \frac{(AB)^{\cancel{2}}}{\cancel{X^2}}\exp(-(\log A)^{\varepsilon})$

<u>THEOREM</u> (MATOMAKI, R.) LET $H=X^{\varepsilon}$. THEN

$$\frac{1}{H}\sum_{x\le n\le x+H}\lambda(n)=o(1)\quad \text{FOR ALMOST ALL } x\in[X,2X].$$

<u>PROOF:</u> BY PARSEVAL

$$\frac{1}{X}\int_{X}^{2X}\Big|\frac{1}{H}\sum_{x\le n\le x+H}\lambda(n)\Big|^2dx \ll \frac{1}{X^2}\overbrace{\int_{(\log X)^A}^{X/H}\Big|\sum_{X\le n\le 2X}\lambda(n)\,n^{it}\Big|^2dt}^{(**)}$$

FIX $P<Q$ SUCH THAT $\displaystyle\sum_{P\le p\le Q}\frac{1}{p}\to+\infty$.

(# OF INTEGERS WITHOUT A PRIME FACTOR IN $[P,Q]$ IS

$$\ll X\prod_{P\le p\le Q}\left(1-\frac{1}{p}\right)=o(X)$$

$$\sum_{X \le n \le 2X} \lambda(n)\, n^{it} = \sum_{\Delta \log P \le k \le \Delta \log Q} P_k(t)\, Q_k(t) + E(t) + D_\Delta(t)$$

$$E(t) = \sum_{\substack{X \le n \le 2X \\ n \text{ HAS NO PRIME} \\ \text{FACTOR IN } [P,Q]}} \lambda(n) .\, n^{it} \qquad P_k(t) = \sum_{\substack{e^{k/\Delta} \le p \le e^{(k+1)/\Delta} \\ P \le p \le Q}} p^{+it} \lambda(p)$$

$$Q_k(t) = \sum_{Xe^{-(k+1)/\Delta} \le m \le 2Xe^{-k/\Delta}} \frac{m^{it} \lambda(m)}{\#\{p \mid m : P \le p \le Q\} + 1}$$

$$D_\Delta(t) = \sum_{Xe^{-1/\Delta} \le m \le X} a(m)\, m^{it} + \sum_{2X \le m \le 2Xe^{1/\Delta}} a(m)\, m^{it}$$

$$(**) \ll \Delta \log \frac{Q}{P} \sum_{\Delta \log P \le k \le \Delta \log Q} \frac{1}{X^2} \int_{(\log X)^A}^{X/H} |P_k(t)\, Q_k(t)|^2\, dt$$

$$+ \frac{1}{X^2} \int_0^{X/H} |D(t)|^2\, dt + \frac{1}{X^2} \int_0^{X/H} |E(t)|^2\, dt$$

$$\int_{-T}^{T} \Big| \sum a(n)\, n^{it} \Big|^2 dt = \big(2T + O(N)\big) \sum_{N \le n \le 2N} |a(n)|^2$$

$$\sum_{P \le p \le Q} \frac{1}{p} \to +\infty$$

$$\int_0^{X/H} |E(t)|^2 dt \le \left(\frac{X}{H} + X \right) \sum_{\substack{n \text{ HAS} \\ \text{NO PRIME} \\ \text{FACTOR IN } [P,Q]}} 1 \ll X^2 \prod_{P \le p \le Q} \left(1 - \frac{1}{p}\right) = o(X^2)$$

$$\int_0^{X/H} |D(t)|^2 \ll \frac{X^2}{\Delta} = o(X^2) \quad \text{if } \Delta \to +\infty$$

$$(I) \quad \frac{1}{X^2} \int_{(\log X)^A}^{X/H} |P_k(t)\, Q_k(t)|^2\, dt$$

BY PNT $\quad |P_k(t)| \leq e^{k/\Delta} \exp(-(\log P)^{\varepsilon})$

PROVIDED THAT $\quad P > \exp((\log X)^{\frac{2}{3}+\varepsilon})$

$$(I) \leq \frac{e^{2k/\Delta}}{X^2}\, e^{-(\log P)^{\varepsilon}} \int_0^{X/H} |Q_k(t)|^2\, dt$$

$$\leq \frac{e^{2k/\Delta}}{X^2}\, e^{-(\log P)^{\varepsilon}} \left(\frac{X}{H} + \frac{X}{e^{k/\Delta}}\right) \frac{X}{e^{k/\Delta}}$$

$$(**) \leq \Delta \log \frac{Q}{P} \sum_{\Delta \log P \leq k \leq \Delta \log Q} \frac{e^{2k/\Delta}}{X^2} \left(\frac{X}{H} + \frac{X}{e^{k/\Delta}}\right) \frac{X}{e^{k/\Delta}}\, e^{-(\log P)^{\varepsilon}}$$

$$+ o(1)$$

$$\leq \Delta \log Q \cdot \frac{Q}{H}\, e^{-(\log P)^{\varepsilon}} + (\Delta \log Q)^2\, e^{-(\log P)^{\varepsilon}} + \underbrace{o(1)}_{\text{“}\frac{1}{\Delta} + \prod_{P \leq p \leq Q}\left(1 - \frac{1}{p}\right)\text{”}}$$

PICK $\quad Q = H \quad (= X^{\varepsilon})$

$\Delta \to +\infty \quad$ VERY SLOWLY

$P = \exp((\log X)^{\frac{2}{3}+\varepsilon})$