ℓ-ADIC TRACE FUNCTIONS IN ANALYTIC NUMBER THEORY

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q $\quad \ell \neq q$ $\quad U \subset \mathbb{P}^1_{\mathbb{F}_q}$, $\mathcal{F}$ lisse on U

$\rho_{\mathcal{F}} : \mathrm{Gal}(K^{sep}/\mathbb{F}_q(x)) \to GL(V)$

= UNRAMIFIED ON U

- $x \in U(\mathbb{F}_q) \longrightarrow K_{\mathcal{F}}(x) = \mathrm{tr}(\mathrm{Frob}_x \mid V)$

$n \geq 1$ $\quad x \in U(\mathbb{F}_{q^n}) \to K_{\mathcal{F},n}(x) = \mathrm{tr}(\mathrm{Frob}_{x,n} \mid V)$

RMQ IF $x_{\mathbb{F}_{q^n}}$ COME FROM $\mathbb{F}_q$

$K_{\mathcal{F},n}(x) = \mathrm{tr}(\mathrm{Frob}^n_{x,\mathbb{F}_q} \mid V)$

THESE $\overline{\mathbb{Q}_\ell}$ - VALUED FCT CAN BE SEEN AS $\mathbb{C}$-VALUED BY CHOOSING $\overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$.

PURITY: $w \in \mathbb{Z}$, $\mathcal{F}_{|U}$ IS PURE OF wgt w

IF FOR ANY $x \in U(\mathbb{F}_{q^n})$ THE Frob_x eigenvalues HAVE MODULUS $(q^n)^{w/2}$

$\Rightarrow |K_{\mathcal{F},n}(x)| \leq (\dim V)(q^n)^{w/2}$

$\dim V = \mathrm{rk}(\mathcal{F})$

$\mathcal{F}$ IS MIXED OF wt $\leq w$ if is a successive ext. OF PURE SHEAVES OF wt $w' \leq w$

Rmq: by tensoring with a (Tate twist), a PURE SHEAF OF WEIGHT w CAN BE MADE OF wt 0

WE WILL ALWAYS NORMALIZE SHEAVES TO BE wt 0.

RMQ: $x \notin U(\mathbb{F}_q)$ $\quad K_{\mathcal{F}}(x) := tr(Frob_x \mid V^{I_x})$

DELIGNE: IF $\mathcal{F}$ IS PURE OF wt 0 THE EIGENVALUE ON $(Frob_x \mid V^{I_x})$ ARE OF $wt \leq 0$

SUMMING TRACE FUNCTIONS OVER $\mathbb{F}_q = \mathbb{A}^1(\mathbb{F}_q)$

THEOREM (GROTHENDIECK - LEFSHETZ TRACE FORMULA)

$\mathcal{F} | U$

$$\sum_{x \in U(\mathbb{F}_q)} K_{\mathcal{F}}(x) = \sum_{i=0}^{2} (-1)^i \, tr(Frob_q \mid H^i_c(U_{\overline{\mathbb{F}}_q}, \mathcal{F}))$$

(↙ H^i_c) COMPACTLY SUPPORTED ETALE COHOM GROUPS OF $\mathcal{F}$

$\forall n \geq 1$

$$\sum_{x \in U(\mathbb{F}_{q^n})} K_{\mathcal{F},n}(x) = \sum_{i=0}^{2} (-1)^i \, tr(Frob_{q^n} \mid H^i_c(\quad))$$

$$- H^0_c = \begin{cases} 0 & \text{IF } U \neq \mathbb{P}^1_{\mathbb{F}_q} \\ V^{G^g} & \text{IF } U = \mathbb{P}^1_{\mathbb{F}_q} \end{cases}$$

$\dim H^0_c \leq rk(\mathcal{F})$

$- H^2_c \simeq V_{G^g}$ = GEOMETRIC CO-INVARIANTS

$\dim H^2_c \leq \dim V$

THEOREM: (GROTH. - OGG - SHAFAREVICH)

$$\chi(U|\mathcal{F}) = \sum_{i=0}^{2} (-1)^i \dim H^i_c$$

$$= rk(\mathcal{F})(2 - |\mathbb{P}^1(\overline{\mathbb{F}}_q) - U(\overline{\mathbb{F}}_q)|$$

$$- \sum_{x \in \mathbb{P}^1 - U(\overline{\mathbb{F}}_q)} swan_x(\mathcal{F})$$

$1 \to P_x \to I_x \to D_x$

$$c(\mathcal{F}) = \mathrm{rk}(\mathcal{F}) + |\mathbb{P} - U(\overline{\mathbb{F}_q})| + \sum_{x \in \mathbb{P}^1 - U} \mathrm{swan}_x(\mathcal{F})$$ = the conductor of $\mathcal{F}$

WE WILL DEAL WITH SHEAVES SUCH THAT $C(\mathcal{F}) \leq C$

WHILE $q \to \infty$.

<u>EXAMPLES</u>: $\chi : \mathbb{F}_q^* \to \mathbb{C}^*$

		rk		
KUMMER :	$\mathcal{L}_\chi$:	1	$U = \mathbb{G}_m$ $c(\chi) = 3$	$Sw_0 = Sw_\infty = 0$
ARTIN - SCHREIER	$\mathcal{L}_{e_q}$	1	$U = \mathbb{A}^1$	$Sw_\infty = 1$
	Kl_2	2	$U = \mathbb{G}_m$	$Sw_0 = 0$ $Sw_\infty = 1$
	Kl_R	R	—	

DELIGNE THM ON wt: $\mathcal{F}$ wt 0

THE FROB_q EIGENVALUE ACTING ON H^1_c ARE OF MODULUS $q^{1/2}$

$$\sum_{x \in U(\mathbb{F}_q)}^{U \neq \mathbb{P}^1} K_{\mathcal{F}}(x) = \mathrm{tr}(\mathrm{Frob}_q \mid H^2_c) + O_{c(\mathcal{F})}(q^{1/2})$$

IN PARTICULAR IF $V_{G^r} = \{0\}$

$$\sum K_{\mathcal{F}}(x) = O(q^{1/2})$$

$\mathcal{F}$ PURE wt 0 THE $\rho_{\mathcal{F}} |_{G^r}$ isss CAN BE DECOMPOSED

INTO A $\oplus$ OF IRR REPS → Geom isotypic

$\Rightarrow \rho_{\mathcal{F}}^{ss} = \bigoplus \rho_{\mathcal{F}_i}$ ← irred $/G^a$ → Induced from $\mathrm{Gal}(K^{sep}/\mathbb{F}_{q^n}(X))$, $n > 1$

IF $\mathcal{F}$ IS GEOM ISOTYPIC

$H_c^2 \neq 0$ IFF $\mathcal{F}$ GEOM IS A SUM OF THE TRIVIAL REP

IN THE LATTER CASE, $\exists\, \alpha_1, \ldots, \alpha_{n^2}$ OF MODULUS 1

SUCH THAT $tr(\mathrm{Frob}_q \mid H_c^2) = q(\alpha_1 + \cdots + \alpha_{n_c^2})$

$$\overline{\lim_{n\to\infty}} \frac{1}{q^n} \sum_{x \in U(\mathbb{F}_{q^n})} K_{\mathcal{F},n}(x) = h_c^2$$

<u>EXERCISE:</u> Kl_2 $\mathbb{F}_{q^n}$ $e_q \longleftrightarrow e_{q,n} = e_q(tr_{\mathbb{F}_{q^n}/\mathbb{F}_q})$