THE KUZNETSOV FORMULA, KLOOSTERMANIA AND APPLICATIONS

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KUZNESOY FORMULA

RECALL $S_K(q,\chi)$ SPACE OF CLASSICAL HOL. MOD. FORMS FOR $\Gamma_0(q)$, CENTRAL CHAR χ (CUSPIDAL) WT $K \geq 2$.

FOR ANY $f \in S_K(q,\chi)$ HAS A FOURIER EXPANSION

$$f(z) = \sum_{n \geq 1} a_f(n)\, e(nz)$$

AIM: PETERSSON FORMULA: IF $B_K(q,\chi)$ IS AN O.N. BASIS FOR $S_K(q,\chi)$:

$$\frac{\Gamma(K-1)}{(4\pi\sqrt{mn})^{K-1}} \sum_{f \in B_K(q,\chi)} \overline{a_f(m)}\, a_f(n)$$

$$= \delta_{n=m} + 2\pi i^{-K} \sum_{\substack{c>0 \\ c \equiv 0 (q)}} \frac{S_\chi(m,n,c)}{c} J_{K-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

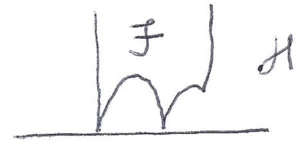
FOR EACH m, $a_f(m)$ IS A LINEAR FUNCTION ON $S_K(q,\chi)$

SO $\exists\, P_m \in S_K(q,\chi)$ SUCH THAT $\langle f, P_m\rangle = a_f(m)$. CAN GIVE EXPLICIT EXPRESSION FOR POINCARÉ SERIES $P_m(z)$

$$\langle f, g\rangle = \int_{\Gamma_0(q)\backslash H} f(z)\overline{g(z)}\, y^k \frac{dx\,dy}{y^2}$$

IN THIS LECTURE $\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \;\middle|\; c \equiv 0 (q) \right\} / \{\pm Id\}$

A FUNDAMENTAL DOMAIN FOR $\Gamma_0(q) \backslash H$ TOUCHES ∂H AT FINITELY MANY POINTS. WE CALL THESE CUSPS



LET $\Gamma_\infty \subseteq \Gamma_0(q)$ BE THE STABILIZER OF ∞.

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

mth HOLOMORPHIC POINCARÉ SERIES @ ∞

$$P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \frac{\overline{\chi(\gamma)}\, e(m \gamma z)}{(cz+d)^k}$$

CHECK: IF $k > 2$ $P_m(z)$ CONVERGES ON $C \subseteq \mathcal{H}$

ASSUME IN THIS TALK $k > 2$.

$P_m(\gamma z) = \chi(\gamma)\,(cz+d)^k\, P_m(z)$

$P_m \in S_k(q, \chi)$

$$\chi(\gamma) = \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\text{def}}{=} \chi(d)$$

COMPUTE $\langle f, P_m \rangle$ BY UNFOLDING

$$= \int_{\Gamma_0(q) \backslash \mathcal{H}} f(z) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \frac{\chi(\gamma)\, e(-m \gamma \bar{z})}{(\overline{cz+d})^k}\, y^k \frac{dx\,dy}{y^2} \quad (*)$$

NOTE: LET $j(\gamma, z) = cz+d$ FOR $\gamma \in SL_2\mathbb{Z}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $z \in \mathcal{H}$

THEN $j(\gamma_1, \gamma_2 z)\, j(\gamma_2, z)$

$$(*) = \int_{\Gamma_\infty \backslash \mathcal{H}} f(z)\, e(-m\bar{z})\, y^k \frac{dx\,dy}{y^2}$$

$$= \int_0^\infty \int_0^1 \sum_{n \geq 1} a_f(n)\, e(nz)\, e(-m\bar{z})\, x\, y^k \frac{dx\,dy}{y^2}$$

$$= \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\, a_f(m)$$

"INCOMPLETE POINCARÉ SERIES"

LET $F(z)$ A FCN ON $\mathcal{H}$ SUCH THAT

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \frac{\overline{\chi(\gamma)}\; F(m\gamma z)}{(cz+d)^k} = P_{F,m}(z) \quad \text{CONVERGES.}$$

EXERCISE:

$$\Gamma_0(q) = \Gamma_\infty \cup \bigcup_{\substack{c>0 \\ c\equiv 0\,(q)}} \; \bigcup^*_{d(c)} \Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_\infty$$

WHERE FOR EACH c, d PICK ANY a, b SUCH THAT $ad - bc = 1$.

ASSUME $F(z+t) = F(z)e(t)$ FOR ANY $t \in \mathbb{R}$.

$$\gamma z = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}$$

$$\Rightarrow F(m\gamma z) = e\left(\frac{ma}{c}\right) F\left(\frac{-mz}{c(cz+d)}\right)$$

$$P_{F,m}(z) = F(mz) + \sum_{c\equiv 0(q)} \sum^*_{d(c)} \sum_{n\in\mathbb{Z}} \frac{\overline{\gamma(d)}\, e\left(\frac{ma}{c}\right) F\left(\frac{-mz}{c(cz+cn+d)}\right)}{(cz+cn+d)^k}$$

$$\sum_{n\in\mathbb{Z}} \frac{F\left(\frac{-mz}{c(cz+cn+d)}\right)}{(cn+cz+d)^k} = \sum_{n\in\mathbb{Z}} \int_{\mathrm{Im}(w)=\mathrm{Im}(z)} \frac{F\left(-\frac{m}{c^2 w}\right) e(-nw)}{(cw)^k}\, dw \; e\left(nz + \frac{nd}{c}\right)$$

$$\Rightarrow P_{m,F}(z) = F(mz) + \sum_{n\in\mathbb{Z}} e(nz) \frac{\sum_{c\equiv 0(q)} \sum_{d(c)} \overline{\chi}(d)\, e\left(\frac{ma+nd}{c}\right)}{c} \int_{\mathrm{Im}\,w = \mathrm{Im}\,z} \frac{F\left(-\frac{m}{c^2 w}\right) e(-nw)}{(cw)^{k-1}} \frac{dw}{w}$$

NOTE: $S_\chi(m,n,c)$ APPEARS IN THIS EXPANSION EVEN WITHOUT PICKING F.

PICK $F(z) = e(z)$. WRITE $P_m(z) = \sum_n a_{P_m}(n) e(nz)$

ASSUME $m \geq 1$

$$a_{P_m}(n) = \begin{cases} \delta_{m=n} + \sum\limits_{c \equiv 0 (q)} \dfrac{S_\chi(m,n,c)}{c} \displaystyle\int_{\substack{\mathrm{Im}(\omega) \\ = \mathrm{Im}(z)}} \dfrac{e\left(\frac{-m}{c^2\omega} - n\omega\right)}{(c\omega)^{k-1}} \dfrac{d\omega}{\omega}, & \text{if } n \geq 1 \\ 0 \quad \text{(shift countour up)}, & \text{if } n \leq 0 \end{cases}$$

PROOF OF PETERSSON FORMULA:

LET P_m, Q_n TWO POINCARÉ SERIES.

COMPUTE $\langle P_m, Q_n \rangle$ IN TWO WAYS.

LHS: EXPAND P_m IN AN ON BASIS $\mathcal{B}_k(q,\chi)$ OF $S_k(q,\chi)$.

$$\langle P_m, Q_n \rangle = \sum_{f \in \mathcal{B}_k(q,\chi)} \langle P_m, f \rangle \langle f, Q_n \rangle$$

RHS: $\langle P_m, Q_n \rangle$ IS THE nth F.C. OF P_m.

qed

LET $\Delta = -y^2 \left(\dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial x^2} \right)$ LAPLACIAN FOR $\mathcal{H}$

$\Gamma = \Gamma_0(q)$ $\quad s \in \mathbb{C}$

LET $A_s(\Gamma \backslash \mathcal{H}) = \left\{ f : \mathcal{H} \to \mathbb{C} \;\middle|\; \begin{array}{l} f(\gamma z) = f(z) \\ \Delta f = s(1-s) f \end{array} \right\}$

KUZRETSOV FORMULA: GENERALIZATION OF PETERSSON FORMULA FOR $A_s(\Gamma \backslash \mathcal{H})$

WEIGHT $k = 0$, TRIVIAL CENTER CHARACTER

INNER PRODUCT: $\langle f, g \rangle = \displaystyle\int f(z) \overline{g(z)} \frac{dx\,dy}{y^2}$

WRITE $s(1-s) = \dfrac{1}{4} + t^2 = \lambda$

$\Delta f = \lambda f$

LEMMA: LET $f \in \mathcal{A}_s(\Gamma \backslash \mathcal{H})$ SUCH THAT $f(iy) = o(e^{2\pi y})$

THEN $\exists\, a, b \in \mathbb{C}$ (a_n) SUCH THAT

$$f(z) = a y^{\frac{1}{2}+it} + b y^{\frac{1}{2}-it} + \sum_{n \in \mathbb{Z}} a_n K_{2it}((2\pi/n) y)\, e(nx)$$

TAKE $P_{F,n}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} F(m\gamma z)$.

WITH $k=0$

χ trivial

SUPPOSE $F(z) = p(\mathrm{Im}\, z)\, e(nz)$

PICK $p(y)$ TO BE AN EIGENFCT OF Δ.

EASIEST CHOICE: $\Delta y^s = s(1-s)\, y^s$

WRITE $P_{F,m}(z) = P_m(z,s)$ WITH THIS CHOICE

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} (m \,\mathrm{Im}\, \gamma z)^s\, e(m\gamma z)$$

EXACTLY AS BEFORE:

$$\langle f, P_m(\cdot, s) \rangle = 2\pi \frac{\Gamma(s-\frac{1}{2}+it)\,\Gamma(s-\frac{1}{2}-it)}{\Gamma(s)}\, a_m$$

$$P_m(z,s) = e(mz) + \sum_{n \neq 0} e(nz) \sum_{c \equiv 0 (q)} \frac{S(m,n,c)}{c^{2s}} \int_{\mathrm{Im}\, w = y} |w|^{-2s}\, e\left(\frac{-m}{c^2 w} - nw\right) dw$$

COMPUTE $\langle P_m(s), Q_n(\cdot, s) \rangle$ IN TWO WAYS TO PROVE

KUZNETSOV FORMULA

LET $u_1, u_2, u_3, \ldots$ BE A COMPLETE ON set FOR

$L_0^2(\Gamma \backslash \mathcal{H})$

← CUSPIDAL SUBSPACE

LET $E_a(z, \frac{1}{2}+it)$ BE THE EISENSTEIN SERIES AT

CUSP a OF EVAL $\frac{1}{4}+t^2$

WRITE $\rho_j(n)$ FOR THE FOURIER COEFF'S OF u_j

$\tau_a(n,t)$ " " $E_a(n, \frac{1}{2}+it)$

THEOREM (RRAGGOMEN - KUZNETSOV FORMULA)

LET $h(t)$ BE ANY FUNCTION SATISFYING:

- $h(t) = h(-t)$
- $h(t)$ HOLOMORPHIC IN $|\mathrm{Im}\, t| \leq \frac{1}{2}+\delta$
- $h(t) \ll (1+|t|)^{-(2+\delta)}$ $\quad \delta > 0$

THEN

$$\sum_{j=1}^{\infty} \overline{\rho_j(m)}\, \rho_j(n) \frac{h(t_j)}{\cosh(\pi t_j)} + \sum_{a \text{ CUSPS}} \frac{1}{4\pi} \int_{\mathbb{R}} \overline{\tau_a(m,t)}\, \tau_a(n,t) \frac{h(t)}{\cosh \pi t}\, dt$$

$$= \delta_{m=n} \left(\frac{1}{\pi^2} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r)\, dr \right) + \sum_{c \equiv 0 (q)} \frac{S(m,n,c)}{c}\, g\left(\frac{4\pi\sqrt{mn}}{c} \right)$$

WHERE $g(x) = \dfrac{2i}{\pi} \displaystyle\int_{-\infty}^{\infty} J_{2it}(x) \frac{t h(t)}{\cosh(\pi t)}\, dt$