LARGE FIXED ORDER CHARACTER SUMS

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χ PRIMITIVE CHAR. MOD q

$$m(\chi) = \max_{t \leq q} \left| \sum_{n \leq t} \chi(n) \right|$$

TRIVIAL $m(\chi) \leq q$.

PÓLYA–VINOGRADOV (1918):

$$m(\chi) \leq C_0 \sqrt{q} \log q$$

MONTGOMERY–VAUGHAN (1977): GRH

$$m(\chi) \leq C_1 \sqrt{q} \log\log q$$

PALEY (1932):

$\exists \infty$ QUAD. χ MOD q:

$$m(\chi) \geq \tilde{C}_1 \sqrt{q} \log\log q$$

GRANVILLE–SOUND (2007):

* GRH

$$C_1 = \begin{cases} \dfrac{2e^{\gamma}}{\pi} & \chi \text{ ODD } (\chi(-1) = -1) \\[2ex] \dfrac{2e^{\gamma}}{\pi\sqrt{3}} & \chi \text{ EVEN } (\chi(-1) = 1) \end{cases}$$

* Ω - RESULT

γ = EULER CONSTANT

$\exists \infty$ χ ODD $\quad \tilde{C}_1 = \dfrac{e^{\gamma}}{\pi}$

$\exists \infty$ χ EVEN $\quad \tilde{C}_1 = \dfrac{e^{\gamma}}{\pi\sqrt{3}}$

* CONJECTURE (G-So):

best possible

Bo - Go - G - Ko (2014):

DISTRIBUTION OF $m(\chi)$

$\chi \bmod q$, q PRIME

BOBER - GOLDMAKHER (2014)

n_p: LEAST QUAD. NONRES MOD p

BURGESS: $n_p \leq p^{\frac{1}{4\sqrt{e}}+o(1)}$

VINOGRADOV CONJ: $n_p \ll p^{\varepsilon}$

GRH: $n_p \leq (\log p)^2$

Bo - Go: BEST KNOWN $C_0 \Rightarrow$ BUGESS

(HILDEBRAND)

* $C_0 \to 0$ AS $q \to \infty$ $\Rightarrow$ VINOGRADOV

* Gr - So CONJECTURE $\Rightarrow$ $n_p \leq (\log p)^{1.37+o(1)}$

C_1

ORD $\chi = n$ $\Leftrightarrow$ $\chi^n = 1$, n SMALLEST

EVEN ORDER CHARACTERS:

GRANVILLE - SOUND (2007)

$k \in \mathbb{N}$: GRH

$\exists\ \infty$ χ ORD $\chi = 2k$

$$m(\chi) \geq \tilde{C}_k \sqrt{q} \log\log q$$

GOLDMAKHER - L (2014): UNCONDITIONAL

PALEY $\tilde{C}_k \to 0$ AS $k \to \infty$

L (2016)

$L(1,\chi)$

* $\tilde{C}_k = \begin{cases} \frac{e^{\gamma}}{\pi}, & \chi \text{ ODD} \\ \frac{e^{\gamma}}{\pi\sqrt{3}}, & \chi \text{ EVEN} \end{cases}$

* $k \leq (\log q)^A$

ODD ORDER CHARACTERS

$g \geq 3$ ODD

$$Q := \begin{cases} q & \text{UNCOND} \\ \log q & \text{GRH} \end{cases}$$

GRANVILLE – SOUND (2007)

$$m(\chi) \ll \sqrt{q}\,(\log Q)^{1-\frac{\delta_g}{2}+o(1)}$$

$$\delta_g := 1 - \frac{g}{\pi} \sin\frac{\pi}{g} > 0$$

GOLDMAKHER (2012):

A) $m(\chi) \ll \sqrt{q}\,(\log Q)^{1-\delta_g+o(1)}$

B) GRH : $\exists \infty$ χ ORD g

$$m(\chi) \gg \sqrt{q}\,(\log\log q)^{1-\delta_g-\varepsilon}$$

GOLDMAKHER – L (2012) : B) UNCONDITIONAL

L – MANGEREL (2017):

$\log_j$ = j^{th} ITERATE OF $\log$

THEOREM 1: (UPPER BOUNDS) ord $\chi = g$

$$* \; m(\chi) \ll \sqrt{q}\,(\log q)^{1-\delta_g}(\log_2 q)^{-\frac{1}{4}+\varepsilon}$$

* SIEGEL ZEROS

NO SIEGEL * $(\log_2 q)^{\varepsilon} \to (\log_3 q)^{O(1)}$

* GRH :

$$m(\chi) \ll \sqrt{q}\,(\log_2 q)^{1-\delta_g}(\log_3 q)^{-1/4}(\log_4 q)^{O(1)}$$

THEOREM 2: $\exists \infty$ χ ORD g :

$$m(\chi) \gg \sqrt{q}\,(\log_2 q)^{1-\delta_g}(\log_3 q)^{-\frac{1}{4}}(\log_4 q)^{O(1)}$$

PÓLYA :

$$\sum_{n \le \alpha q} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{\substack{1 \le |n| \\ \le q}} \frac{\bar{\chi}(n)}{n} \left(1 - e(-n\alpha)\right) + O(\log q)$$

$$e(x) = e^{2\pi i x}$$

$$\tau(\chi) = \sum_{n \bmod q} \chi(n)\, e\left(\frac{n}{q}\right)$$

χ PRIMITIVE , $|\tau(\chi)| = \sqrt{q}$

$$m(\chi) \longleftrightarrow \sum_{|n| \le q} \frac{\chi(n)}{n} e(n\alpha)$$

<u>Gr-Sound</u> :

GRH

$$\sum_{|n| \le q} \frac{\chi(n)}{n} e(n\alpha) = \sum_{\substack{|n| \le q \\ p \mid n \Rightarrow p \le (\log q)^A}} \frac{\chi(n)}{n} e(n\alpha)$$

$$+ o(1)$$

$$\exp\left(\sum_{p \le (\log q)^A} \frac{1}{p}\right) \ll \log_2 q$$

CIRCLE METHOD $\rightarrow$ $\alpha \approx \frac{b}{m}$

$$\alpha = \frac{b}{m}$$

$$\sum_{|n| \le q} \frac{\chi(n)}{n} e\left(\frac{nb}{m}\right) = \frac{1}{\phi(m)} \sum_{\psi \bmod m} \bar{\psi}(b)\tau(\psi) \times$$

$$\chi\bar{\psi}(-1) = -1 \quad \Leftarrow \quad \sum_{|n| \le q} \frac{\chi\bar{\psi}(n)}{n}$$

UPPER BOUNDS FOR $m(\chi)$:

MONTGOMERY – VAUGHAN:

$\alpha \in$ MINOR ARC (*) SMALL

$\alpha \in$ Major ARC: $\alpha \approx \frac{b}{m}$, $m \le (\log q)^A$

Gr - So: $(*) \approx \sum_{|n| \le N} \frac{\chi(n)}{n} e\left(\frac{nb}{m}\right)$

$N = N(\alpha, b, m, q)$

$\rightarrow \sum_{n \le N} \frac{\chi\overline{\psi}(n)}{n}$ $\qquad \chi\overline{\psi}(-1) = -1$

ψ mod m, m "small"

Gr-So:

I) * $\sum \frac{\chi\overline{\Psi}(n)}{n} \ll (\log x)\exp\left(-\frac{\mathbb{D}^2(\chi, \psi, x)}{2}\right)$

$$\mathbb{D}^2(\chi, \psi, x) = \left(\sum_{p \le x} \frac{1 - \operatorname{Re} \chi\overline{\psi}(p)}{p}\right)^{1/2}$$

II) * REPULSION: ONLY ONE ψ MATTERS

III) * $\mathbb{D}^2(\chi, \psi, x) \ge (\delta_g + o(1)) \log_2 x$

GOLDMAKHER 2012:

LOGARITHM HÁLASZ:

MV, TENENBAUM: $\qquad f$ COMP $\quad |f| \le 1$

$$\sum_{n \le x} \frac{f(n)}{n} \ll (\log x)\exp(-m(f, x, T)) + \frac{1}{T} \qquad T \ge 1$$

$$m(f, x, T) = \min_{|t| \le T} \mathbb{D}^2(f, n^{it}, x)$$

III)* $m(\chi \overline{\psi}, x, T) \ge (\delta_g + o(1)) \log_2 x$

$1 \le T \le (\log x)^2$

L-M ANGEREL:

III) $T \le (\log x)^{-\theta}$ $\qquad k = \mathrm{ord}\, \psi$

$$m(\chi \overline{\psi}, x, T) \ge \left(\delta_g + \frac{c}{k^2}\right) \log_2 x$$

SIGGEL $\longleftarrow$ $- \varepsilon \log m + o(\log_2 m)$

* GRH :

$$m(\chi\overline{\psi}, x, T) \le \delta_g \log_2 x + O(\log_2 m)$$

$T \ge 1$

n^{it} $\qquad \frac{1}{x} \sum_{n \le x} n^{it} \sim \frac{x^{it}}{1+it}$

$$\sum_{n \le x} \frac{n^{it}}{n} \asymp \min\left(\frac{1}{|t|}, \log x\right)$$

small unless t "SMALL"

<u>THEOREM 3 (L-M ANGEREL)</u> $\qquad 0 \le T < 1$

$$\sum \frac{f(n)}{n} \ll \log x \, e^{-m(f, x, T)} + \frac{1}{T}$$

COR: $\sum \frac{f(n)}{n} \gg (\log x)^{\theta}$

$\Rightarrow f \approx n^{it}$ FOR SOME $|t| \ll (\log x)^{-\theta}$

Ω RESULTS : THEOREM 2

GOLD - L :

$$\max_{\alpha} \left| \sum_{n \le q} \frac{\chi(n)}{n} e(n\alpha) \right| \xrightarrow[\text{FOURIER ANALYSIS}]{\text{PALEY}}$$

$$\max_{\alpha} \left| \sum_{n \le \log q} \frac{\chi(n)}{n} e(n\alpha) \right|$$

$\alpha = \frac{b}{m}$

$$\downarrow$$

$$\sum_{n \le \log q} \frac{\chi\bar{\psi}(n)}{n} \xrightarrow[\text{loss } (\log_3 q)]{} \mathbb{D}(\chi, \psi, \log q)$$

L-MANGEREL : $\alpha = \frac{b}{m}$

$$\rightarrow \sum_{n \le q} \frac{\chi\bar{\psi}(n)}{n} \rightarrow L(1, \chi\bar{\psi})$$

$$\xrightarrow[\text{"most } \chi\text{"}]{\text{ZERO-DENSITY}} \prod_{p \le \log} \left(\frac{1 - \chi\bar{\psi}(p)}{p} \right) \asymp \log_2 q \, e^{-\mathbb{D}^2}$$

MINIMIZE $\mathbb{D}^2(\chi, \psi, \log q)$ FOR MANY χ