On Epstein's zeta function and related results in the geometry of numbers

Anders Södergren

Department of Mathematical Sciences
Chalmers University of Technology

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The space of lattices

- Let X_n denote the space of n-dimensional lattices of covolume 1.

- We identify X_n with the homogeneous space $\mathrm{SL}(n,\mathbb{Z}) \setminus \mathrm{SL}(n,\mathbb{R})$:

$$\mathbb{Z}^n g \subset \mathbb{R}^n \longleftrightarrow \mathrm{SL}(n,\mathbb{Z})g$$

- We equip X_n with the probability measure μ_n induced from the Haar measure on $\mathrm{SL}(n,\mathbb{R})$.

The Epstein zeta function

- For $L \in X_n$ and $\operatorname{Re} s > \frac{n}{2}$ the **Epstein zeta function** is defined by

$$E_n(L, s) := \sum_{\boldsymbol{m} \in L \setminus \{\boldsymbol{0}\}} |\boldsymbol{m}|^{-2s}.$$

- $E_n(L, s)$ has an analytic continuation to $\mathbb{C} \setminus \{\frac{n}{2}\}$ and satisfies the functional equation

$$F_n(L, s) := \pi^{-s} \Gamma(s) E_n(L, s) = F_n\Big(L^*, \frac{n}{2} - s\Big).$$

(Here $L^* := \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{Z} \quad \forall \boldsymbol{y} \in L\}$ is the dual lattice of L.)

The Epstein zeta function is of interest in...

- **Analytic number theory:** $E_n(L, s)$ is analogous to $\zeta(s)$; $\zeta(2s) = \frac{1}{2}E_1(\mathbb{Z}, s)$. However, RH for $E_n(L, s)$ fails for a generic $L \in X_n$ $(n \geq 2)$.
- **The geometry of numbers:** The lattice sphere packing problem can be formulated as an optimization problem for $E_n(L, s)$.

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- **The geometry of numbers:** The lattice sphere packing problem can be formulated as an optimization problem for $E_n(L,s)$.
- **Automorphic forms:** $E_n(L,s)$ is a maximal parabolic Eisenstein series for $\mathrm{GL}(n,\mathbb{R})$.
- **Algebraic number theory:** A "twisted" version of $E_n(L,s)$ appears in Stark's proof that there exist exactly nine imaginary quadratic fields of class number one.
- **Theoretical physics and chemistry**: $E_n(L,s)$ is related to the electrostatic energy in crystals.

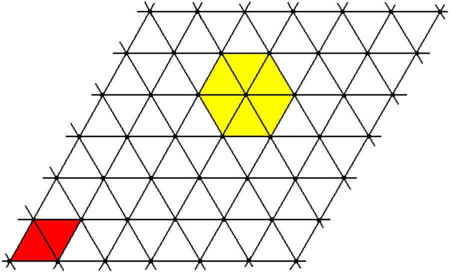
Minima of $E_n(L, s)$

Theorem (Rankin - Cassels - Ennola - Diananda)

Let L_2 denote the hexagonal lattice in X_2. Then

$$E_2(L, s) \geq E_2(L_2, s) \qquad \forall s > 0, \forall L \in X_2,$$

with equality iff $L = L_2$.



Minima of $E_n(L, s)$

Definition

i) For any $n \geq 2$, we let $L_n \in X_n$ denote a lattice giving the densest lattice sphere packing in $\mathbb{R}^n$.

ii) For any $n \geq 2$, we call L_n **universal** if

$$E_n(L, s) \geq E_n(L_n, s) \qquad \forall s > 0, \forall L \in X_n,$$

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Theorem (Sarnak - Strömbergsson)

For $n = 4$, 8 and 24 and $s > 0$, $E_n(L, s)$ has a strict local minimum at $L = L_n$.

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Question 1 (Sarnak - Strömbergsson)

Does there exist arbitrarily large n for which L_n is universal?

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A straightforward averaging argument shows that if L_n is universal then $E_n(L_n, s)$ has no zeros in the interval $(0, \infty)$.

Question 2 (Sarnak - Strömbergsson)

Does there exist, for arbitrarily large n*, a lattice* $L \in X_n$ *for which* $E_n(L, s)$ *has no zeros in the interval* $(0, \infty)$*?*

Minima of $E_n(L,s)$

Question 2 (Sarnak - Strömbergsson)

Does there exist, for arbitrarily large n, a lattice $L \in X_n$ for which $E_n(L,s)$ has no zeros in the interval $(0,\infty)$?

The first step towards an answer to *Question 2* is the following result.

Theorem (Sarnak - Strömbergsson)

If $\varepsilon > 0$ is fixed, then

$$Prob_{\mu_n}\left\{ L \in X_n \;\middle|\; \left| \tfrac{\partial}{\partial s} E_n(L,s)_{|s=0} - (1-\gamma-\log\pi) \right| < \varepsilon \right\} \to 1$$

as $n \to \infty$, where γ is Euler's constant.

Value distribution of $E_n(L, s)$

- Let V_n denote the volume of the unit ball in $\mathbb{R}^n$.
- Let $\mathcal{P} = \{N(V), V \geq 0\}$ be a Poisson process on $\mathbb{R}^+$ with constant intensity $\frac{1}{2}$ and let $R(V) := 2N(V) - V$.

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Theorem 1 (S.)

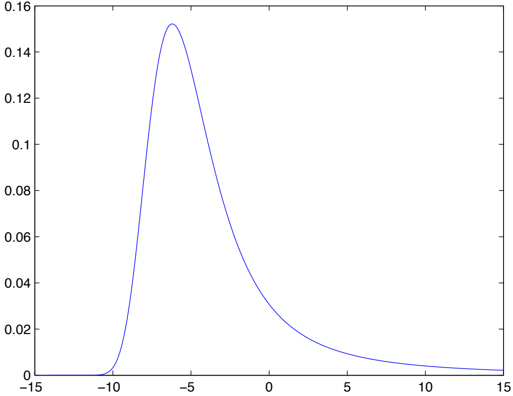
Let $\frac{1}{4} < c_1 < c_2 < \frac{1}{2}$. For each $n \in \mathbb{Z}^+$ consider $c \mapsto V_n^{-2c} E_n(\cdot, cn)$ as a random function in $C([c_1, c_2])$. This random function converges in distribution to

$$c \mapsto \int_0^\infty V^{-2c}\, dR(V)$$

as $n \to \infty$.

A few remarks

- The limit variable is well-defined and for fixed $\frac{1}{4} < c < \frac{1}{2}$ the integral $\int_0^\infty V^{-2c}\, dR(V)$ has a strictly $\frac{1}{2c}$-stable distribution.



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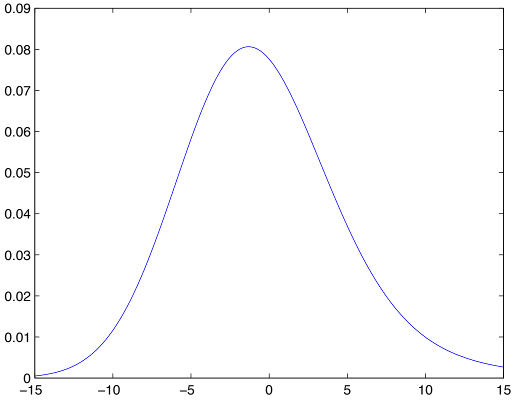
- $(2c - \frac{1}{2})^{\frac{1}{2}} \int_0^\infty V^{-2c}\, dR(V)$ converges in distribution to $N(0,1)$ as $c \to \frac{1}{4}+$.



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- For $c > \frac{1}{2}$, the random variable $V_n^{-2c} E_n(\cdot, cn)$ converges to the distribution of $2\int_0^\infty V^{-2c}\, dN(V) = 2\sum_{j=1}^\infty T_j^{-2c}$ as $n \to \infty$.

- At the moment we do not understand the precise behavior of $E_n(\cdot, cn)$ as $c \to \frac{1}{4}$ and $n \to \infty$.

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Does there exist, for arbitrarily large n, a lattice $L \in X_n$ for which $E_n(L, s)$ has no zeros in the interval $(0, \infty)$?

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Corollary

For any fixed $\frac{1}{4} < c_1 < c_2 \leq \frac{1}{2}$,

$$\lim_{n\to\infty} Prob_{\mu_n}\Big\{L \in X_n \,\Big|\, E_n(s, L) < 0 \text{ for all } s \in [c_1 n, c_2 n] \setminus \{\tfrac{1}{2}n\}\Big\} = Prob\Big\{\int_0^\infty V^{-2c}\, dR(V) < 0 \text{ for all } c \in [c_1, c_2] \setminus \{\tfrac{1}{2}\}\Big\}.$$

Moreover, the above limit $\mathcal{L}$ satisfies $0 < \mathcal{L} < 1$.

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$$= Prob\Big\{ \int_0^\infty V^{-2c}\, dR(V) < 0 \text{ for all } c \in [c_1, c_2] \setminus \{\tfrac{1}{2}\} \Big\}.$$

Moreover, the above limit $\mathcal{L}$ satisfies $0 < \mathcal{L} < 1$.

Theorem 2 (S.)

$Prob_{\mu_n}\{ L \in X_n \,|\, E_n(L,s)$ has a zero in $(0,\infty)\} \to 1$ as $n \to \infty$.

Value distribution of $E_n(L, s)$

- Let V_n denote the volume of the unit ball in $\mathbb{R}^n$.
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Outline of the proof of Theorem 1

We have, for $s \in \mathbb{C} \setminus \{0, \frac{n}{2}\}$,

$$F_n(L,s) = \pi^{-s}\Gamma(s)E_n(L,s) = \Big(-\frac{1}{\frac{n}{2}-s} + \sum_{\boldsymbol{m}\in L\setminus\{\boldsymbol{0}\}} G\big(s, \pi|\boldsymbol{m}|^2\big)\Big) + \Big(-\frac{1}{s} + \sum_{\boldsymbol{m}\in L^*\setminus\{\boldsymbol{0}\}} G\big(\tfrac{n}{2}-s, \pi|\boldsymbol{m}|^2\big)\Big)$$

where

$$G(s,x) := \int_1^\infty t^{s-1}e^{-xt}\,dt, \qquad \operatorname{Re} x > 0.$$

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Let

$$H_n(L,s) := -\frac{1}{\frac{n}{2}-s} + \sum_{\boldsymbol{m}\in L\setminus\{\boldsymbol{0}\}} G\left(s, \pi|\boldsymbol{m}|^2\right).$$

Then

$$F_n(L,s) = H_n(L,s) + H_n(L^*, \tfrac{n}{2}-s).$$

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The analysis of

$$H_n(L,s) := -\frac{1}{\frac{n}{2}-s} + \sum_{\boldsymbol{m}\in L\setminus\{\boldsymbol{0}\}} G\left(s, \pi|\boldsymbol{m}|^2\right)$$

is difficult since we have **exponential cancellation** between the sum and the term $-(\frac{n}{2}-s)^{-1}$:

For any fixed $c \in (\frac{1}{4}, \frac{1}{2})$ there exists $\delta > 0$ such that

$$\mathrm{Prob}_{\mu_n}\left\{L \in X_n \,\middle|\, \left|H_n(L, cn)\right| < e^{-\delta n}\right\} \to 1 \qquad \text{as } n \to \infty.$$

Outline of the proof of Theorem 1

We tackle this problem by writing $H_n(L, cn)$ as an integral,

$$\begin{aligned} H_n(L, cn) &= -\frac{1}{\frac{n}{2} - cn} + \sum_{\boldsymbol{m} \in L \setminus \{\boldsymbol{0}\}} G\left(cn, \pi |\boldsymbol{m}|^2\right) \\ &= -\frac{1}{\frac{n}{2} - cn} + \int_0^\infty G\left(cn, \pi \left(\frac{V}{V_n}\right)^{\frac{2}{n}}\right) dN_n(V) \\ &= \int_0^\infty G\left(cn, \pi \left(\frac{V}{V_n}\right)^{\frac{2}{n}}\right) dR_n(V) \\ &\approx \text{FACTOR}(c, n) \cdot \int_0^\infty V^{-2c} \, dR(V), \end{aligned}$$

for all $\frac{1}{4} < c < \frac{1}{2}$, where $N_n(V) = N_n(L, V)$ equals the number of non-zero lattice points of L in the closed ball of volume V centered at the origin, and $R_n(V) = N_n(V) - V$.

Two main ingredients in the final step above

1. Bound of $R_n(V)$:

Theorem 3 (S.)

For any $n \geq 3$ and for almost every $L \in X_n$, we have $|R_n(V)| \ll_\varepsilon V^{\frac{1}{2}}(\log V)^{\frac{3}{2}+\varepsilon}$ as $V \to \infty$.

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- Note that $R_n(V) = N_n(V) - V$ is the remainder term in the circle problem generalized to dimension n and general ellipsoids.
- The central part of the proof is the variance relation

$$\mathbb{E}\left((R_n(V+\Delta) - R_n(V))^2\right) < 5\Delta,$$

valid for $V \geq 0$, $\Delta > 0$ and $n \geq 3$.

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This bound is proved using Rogers' formula

$$\int_{X_n} \sum_{\boldsymbol{m}_1,\boldsymbol{m}_2 \in L\setminus\{\boldsymbol{0}\}} \rho(\boldsymbol{m}_1,\boldsymbol{m}_2)\, d\mu_n(L)$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho(\boldsymbol{x}_1,\boldsymbol{x}_2)\, d\boldsymbol{x}_1 d\boldsymbol{x}_2 + \frac{2}{\zeta(n)} \sum_{d_1=1}^{\infty}\sum_{d_2=1}^{\infty} \int_{\mathbb{R}^n} \rho(d_1\boldsymbol{x}, d_2\boldsymbol{x})\, d\boldsymbol{x},$$

with ρ a suitable characteristic function on $(\mathbb{R}^n)^2$.

Two of the main ingredients in the final step above

2. The connection between lengths of lattice vectors and the Poisson process $\mathcal{P} = \{N(V), V \geq 0\}$:

- Given $L \in X_n$, order the non-zero vectors by increasing length as $\pm\boldsymbol{v}_1, \pm\boldsymbol{v}_2, \pm\boldsymbol{v}_3, \ldots$; define $\mathcal{V}_j(L) := V_n|\boldsymbol{v}_j|^n$.
- Let $T_1, T_2, T_3, \ldots$ denote the points of the Poisson process $\mathcal{P}$ ordered so that $0 < T_1 < T_2 < T_3 < \cdots$.

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Theorem 4 (S.)

The sequence $\{\mathcal{V}_j(\cdot)\}_{j=1}^{\infty}$ converges in distribution to the sequence $\{T_j\}_{j=1}^{\infty}$ as $n \to \infty$.

Corollary

$R_n(V)$ tends in distribution to $R(V)$ as $n \to \infty$, for any $V \geq 0$.

Outline of the proof of Theorem 1

Recall:

$$H_n(L, cn) = \int_0^\infty G\left(cn, \pi\left(\frac{V}{V_n}\right)^{\frac{2}{n}}\right) dR_n(V)$$
$$\approx \text{FACTOR}(c, n) \cdot \int_0^\infty V^{-2c}\, dR(V),$$

for all $\frac{1}{4} < c < \frac{1}{2}$, where $N_n(V) = N_n(L, V)$ equals the number of non-zero lattice points of L in the closed ball of volume V centered at the origin, and $R_n(V) = N_n(V) - V$.

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To understand the value distribution of $E_n(L, s)$ at the central point, i.e. the distribution of $E_n(L, \frac{n}{4})$, as $n \to \infty$.

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The current work focuses on two key parts:

- **Truncation issues.** Need to understand the limit behavior of the error term in the generalized circle problem in the situation where the size of the ball is growing with the dimension.
- **The joint distribution** of $E_n(L, cn)$ on $c \leq \frac{1}{4}$ and $c \geq \frac{1}{4}$. Need to understand the statistical relation between a random lattice L and its dual L^*.

The generalized circle problem for a random lattice

Recall that above we used the following:

Theorem (S.)

For any $n \geq 3$ and for almost every $L \in X_n$, we have $|R_n(V)| \ll_\varepsilon V^{\frac{1}{2}}(\log V)^{\frac{3}{2}+\varepsilon}$ as $V \to \infty$.

The generalized circle problem for a random lattice

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What is expected?

Conjecture (Götze?)

For any $n \geq 2$ and for almost every $L \in X_n$, we have $|R_n(V)| \ll_{L,\varepsilon} V^{\frac{1}{2}-\frac{1}{2n}+\varepsilon}$ as $V \to \infty$.

The generalized circle problem for a random lattice

In the situation where the size of the ball is growing with the dimension we can prove the following **central limit theorem**:

Theorem (Strömbergsson-S.)

Let $f:\mathbb{Z}^+ \to \mathbb{R}^+$ *be any function satisfying* $\lim_{n\to\infty} f(n) = \infty$ *and* $f(n) = O_\varepsilon(e^{\varepsilon n})$ *for every* $\varepsilon > 0$*. Then*

$$\frac{1}{\sqrt{2f(n)}} R_n(f(n)) \xrightarrow{\text{d}} N(0,1) \qquad \text{as } n \to \infty.$$

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In fact, if for each n *we let* S_n *be a Borel measurable subset of* $\mathbb{R}^n$ *satisfying* $\mathrm{vol}(S_n)=f(n)$ *and* $S_n=-S_n$, *then*

$$\frac{\#(L\cap S_n\setminus\{\mathbf{0}\})-f(n)}{\sqrt{2f(n)}} \xrightarrow{\text{d}} N(0,1) \qquad \text{as } n\to\infty.$$

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We also have the following functional version of our result:

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Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ *be any function satisfying* $\lim_{n\to\infty} f(n) = \infty$ *and* $f(n) = O_\varepsilon(e^{\varepsilon n})$ *for every* $\varepsilon > 0$. *The distribution of the random function*

$$t \mapsto \frac{1}{\sqrt{2f(n)}} R_{n,L}(tf(n)) \qquad \text{(on the interval } [0,1])$$

converges in distribution to one-dimensional Brownian motion.

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Remark

This result is not strong enough to study $E_n(L, \frac{n}{4})$ as $n \to \infty$. At the present preliminary stage we need the above result also for functions $f(n)$ that grows as rapidly as $e^{\frac{1}{2}(1-\log 2)n}$.

Work in progress

Another central problem in our program is to understand the **joint distribution** of the vector lengths of a random lattice $L \in X_n$ and its dual lattice L^* (as $n \to \infty$).

As a first step in this direction we have developed a formula for the expected value of sums on the form

$$\sum_{\boldsymbol{m}_1,\ldots,\boldsymbol{m}_{k_1} \in L} \; \sum_{\boldsymbol{m}_{k_1+1},\ldots,\boldsymbol{m}_{k_1+k_2} \in L^*} f(\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{k_1+k_2}).$$

However, it is not yet clear how to express our formula as explicitly as possible in the case of general k_1 and k_2.

Work in progress

In the special case with $k_1 = k_2 = 1$ and $f(\boldsymbol{m}_1, \boldsymbol{m}_2) = f_1(\boldsymbol{m}_1) f_2(\boldsymbol{m}_2)$ we have the following *explicit* result:

Theorem (Strömbergsson-S.)

Let f_1 and f_2 be Schwartz functions. Then

$$\int_{X_n} \sum_{\boldsymbol{m}_1 \in L} \sum_{\boldsymbol{m}_2 \in L^*} f_1(\boldsymbol{m}_1) f_2(\boldsymbol{m}_2) \, d\mu_n(L) = f_1(\boldsymbol{0}) f_2(\boldsymbol{0}) + \widehat{f_1}(\boldsymbol{0}) f_2(\boldsymbol{0})$$

$$+ f_1(\boldsymbol{0}) \widehat{f_2}(\boldsymbol{0}) + \sum_{k \in \mathbb{Z}} \frac{\sigma_{1-n}(k)}{\zeta(n)} \int_{\mathbb{R}^n} f_1(\boldsymbol{x}) |\boldsymbol{x}|^{-1} \left(\int_{\{\boldsymbol{u} \in \mathbb{R}^n | \langle \boldsymbol{u}, \boldsymbol{x} \rangle = k\}} f_2(\boldsymbol{u}) \, d\boldsymbol{u} \right) d\boldsymbol{x},$$

where

$$\sigma_{1-n}(k) = \sum_{\substack{d|k \\ d>0}} d^{1-n}.$$

Thank you for your attention!