MINI - COURSE ON MULTIPLICATIVE FUNCTIONS

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<ins>THEOREM</ins> : (M. - RADZIWIŁŁ)

AS SOON AS $H \xrightarrow[X \to \infty]{} \infty$, ONE HAS $\sum_{x \leq n \leq x+H} \lambda(n) = o(H)$

FOR ALMOST ALL $x \sim X$.

BY PARSEVAL

$$\frac{1}{X} \int_{X}^{2X} \left| \frac{1}{H} \sum_{x \leq n \leq x+H} \lambda(n) \right|^2 dx$$

$$\ll \frac{1}{X^2} \int_{(\log X)^{100}}^{X/H} \left| \sum_{n < X} \lambda(n) n^{it} \right|^2 dt$$

WANT $o(1)$.

MVT GIVES $\frac{1}{X^2}\left(\frac{X}{H} + X\right) X = O(1)$

CHOOSE $P_j < Q_j$ SUCH THAT $\sum_{P_j < p < Q_j} \frac{1}{p} \to \infty$.

WE USE THE IDENTITY

$$\lambda(n) = \sum_{\substack{pm = n \\ P_j < p < Q_j}} \frac{\lambda(p)\lambda(m)}{\#\{p \mid m : P_j < p < Q_j\} + 1} + \lambda(n) \mathbb{1}_{p \mid n \Rightarrow p \notin [P_j, Q_j]}$$

TO WRITE

$$\sum_{n \sim X} \lambda(n) n^{it} = \sum_{\log P_j \leq \kappa \leq \log Q_j} P_\kappa(t) + Q_\kappa(t) + E_j(t) + D_{\Delta_j}(t)$$

HERE $P_\kappa(t) = \sum_{\substack{e^\kappa < p < e^{\kappa+1} \\ P_j < p < Q_j}} \lambda(p) p^{it}$, $Q_\kappa(t) = \sum_{Xe^{-\kappa-1} \leq m \leq 2Xe^{-\kappa}} \frac{\lambda(m) m^{it}}{\#\{p \mid m \Rightarrow p \in [P_j, Q_j]\} + 1}$

$E_j(t)$, $D(t)$ SMALL ERRORS

NOW, FOR ANY $T \subseteq [(\log x)^{100}, \frac{X}{H}]$ WE HAVE FOR

ALL j

$$I(T) \int_{\gamma} |\sum_{n \sim X} \lambda(n) n^{it}|^2 dt \ll \left(\log \frac{Q_j}{P_j}\right)^2$$

$$\underbrace{\frac{1}{X^2} \int_T |P_{K_j}(t)\, Q_{K_j}(t)|^2 dt}_{I(T,j)} + o(1)$$

FOR SOME $K_j \in [\log P_j, \log Q_j]$

LET $T_1 = \{t \in [(\log X)^{100}, \frac{X}{H}] : |P_{K_1}(t)| \leq e^{K_1(1-\alpha_1)}\}$

$\alpha_1 \in (0, 1/2)$

THEN $I(T_1,1) \ll \frac{1}{X^2} \cdot e^{2K_1(1-\alpha_1)} \int_{(\log X)^{100}}^{X/H} |Q_{K_1}(t)|^2 dt$

$$\ll \frac{1}{X^2} e^{2K_1(1-\alpha_1)} \cdot \left[\frac{X}{H} + \frac{X}{e^{K_1}}\right] \cdot \frac{X}{e^{K_1}}$$

$$\ll \frac{e^{K_1(1-2\alpha_1)}}{H} + e^{-2K_1\alpha_1} = o(1)$$

WHEN $e^{K_1} \leq H$ ie $Q_1 \leq H$

PROBLEM: WHEN H SMALL
SO IS $P_1 < Q_1 < H$,
$\Rightarrow T_1$ NOT EVERYTHING

TAKE $P_1 < Q_1 < H$

PICK $Q_j > P_j > Q_{j-1} > P_{j-1} > \ldots$

SUCH THAT $\prod_{P_j < p < Q_j} \left(1 - \frac{1}{p}\right) \to 0$

DEFINE $T_j = \{t \in [(\log x)^{100}, X/H] : |P_{k_j}(t)| \le e^{k_j(1-\alpha_j)}\} \setminus \bigcup_{i=1}^{j-1} T_j$

$j = 1, \dots, S$ SUCH THAT

$P_j, Q_j \in [\exp((\log x)^{4/5}), \exp((\log x)^{9/10})]$

$$\mathcal{U} = [(\log x)^{100}, X/H] \setminus \bigcup_{j=1}^{S} T_j$$

$$\frac{1}{X^2} \int_{(\log x)^{100}}^{X/H} |\sum f(n) n^{it}|^2 dt = \sum_{j=1}^{S} I(T_j) + I(\mathcal{U})$$

$$\ll \underbrace{\sum_{j=1}^{S} (\log Q_j)^2 \cdot I(T_j, j)}_{o(1)} + \underbrace{(\log Q_j)^2 \cdot I(\mathcal{U}_i, S)}_{+o(1)}$$

WANT $o(1)$

$j = 1$ OK!

FOR $j > 1$, $\quad I(T_j, j) \ll \frac{1}{X^2} e^{2k_j(1-\alpha_j)} \int_{T_j} |Q_{k_j}(t)|^2 dt$

$$\left(\begin{array}{l} \text{MVT GIVES } \frac{1}{X^2} e^{2k_j(1-\alpha_j)} \left(\frac{X}{H} + \frac{X}{e^{k_j}} \right) \frac{X}{e^{k_j}} \\ \qquad = \frac{e^{k_j(1-2\alpha_j)}}{H} + e^{-2\alpha_j k_j} \\ \text{TO WIN NEED } \underbrace{e^{k_j}}_{\in [P_j, Q_j]} < H \quad \text{DOES NOT WORK!} \\ \qquad\qquad\qquad\qquad\qquad Q_{k_j} \text{ TOO SHORT!} \end{array} \right)$$

FOR $t \in T_j$:

$$[|P_{k_{j-1}}(t)| e^{-k_{j-1}(1-\alpha_{j-1})}]^{2A} \ge 1$$

FOR ANY $A > 0$. HENCE

$$I(T_j, j) \ll \frac{1}{X^2} e^{2k_j(1-\alpha_j)} e^{-2Ak_{j-1}(1-\alpha_{j-1})}$$

$$\cdot \int_{T_j} |Q_{k_j}(t)|^2 \cdot |P_{k_{j-1}}(t)|^{2A} dt$$

CHOOSING $A = \left\lceil \frac{K_j}{K_{j-1}} \right\rceil$ MAKES $Q_{K_j}(t) \cdot P_{K_{j-1}}(t)^A$

GOOD FOR MVT

HAVE LENGTH $\underset{\frown}{\smile} X$.

<u>EXERCISE</u>: $$\int_{-X/H}^{X/H} | P_{K_{j-1}}(t)^A \, Q_{K_{j-1}}(t) |^2 \, dt$$

$$\ll (A+1)!^2 \, e^{A + K_{j-1}} \cdot X^2$$

PRETEND $A = \frac{K_j}{K_{j-1}}$. THEN

$$I(T_j, j) \ll e^{2A\log A + K_{j-1} - 2(\alpha_j - \alpha_{j-1}) K_j}$$
$$= e^{K_{j-1} + K_j \left[-2(\alpha_j - \alpha_{j-1}) + \frac{2\log K_j}{K_{j-1}} \right]} = o\left(\frac{1}{j^2 (\log Q_j)^2} \right)$$

IF CHOOSE $\alpha_j = \frac{1}{5} - \frac{1}{40_j}$ $\quad Q_1 = H, \; P_1 = (\log Q_1)^{800}$

$$P_j = \exp(j^{4j} (\log Q_1)^{j-1} \log P_1)$$

$$Q_j = \exp(j^{4j+2} (\log Q_1)^j)$$

---- x ----

NOTE $\mathcal{U} \subseteq \{ t \in [(\log X)^{100}, \frac{X}{H}] : |P_{K_j}(t)| \geq e^{\frac{4}{5} K_s} \}$

<u>EXERCISE</u>: $$|\mathcal{U}| \ll \left(\frac{X}{H} \right)^{\frac{2}{5} + \varepsilon}$$

<u>LEMMA</u> (HALASZ - MONTGOMERY) LET $a_n \in \mathbb{C}$.

LET $\mathcal{T} \subseteq [-T, T]$ WELL-SPACED, $|t_i - t_j| \geq 1 \quad i \neq j$

THEN $$\sum_{t \in \mathcal{T}} \Big| \sum_{n \sim N} a_n n^{it} \Big|^2 \ll (N + \underbrace{|\mathcal{T}| \sqrt{T}}_{T \text{ IN MVT}}) \log T \cdot \sum |a_n|^2$$

$$I(U, S) \ll \frac{1}{X^2} \sup_{(\log X)^{100} \le |t| \le \frac{X}{H}} |P_{K_j}(t)|^2 \cdot \int_U |Q_{K_j}(t)|^2 \, dt$$

$$\ll \frac{1}{X^2} (\log X)^{-200} e^{2K_j} \left(\frac{X}{e^{K_j}} + \left(\frac{X}{H}\right)^{9/10} \right) \frac{X}{e^{K_j}}$$

$$\ll (\log X)^{-200} = o(1)$$