ℓ-ADIC TRACE FUNCTIONS IN ANALYTIC NUMBER THEORY

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$$\frac{1}{q^n} \sum_{x \in V(\mathbb{F}_{q^n})} K_{\mathcal{F},n}(x) = (\alpha_1^n + \dots + \alpha_{k_0 \operatorname{coinv}}^n) + O_{C(\mathcal{F})}(q^{-n/2})$$

CORRELATIONS OF TRACE FUNCTIONS:

K_f K_g $\mathcal{F}, \mathcal{G}$ GEOM ISOTYPIC PURE wt 0

$$\mathcal{C}_n(\mathcal{F}, \mathcal{G}) = \frac{1}{q^n} \sum_{x \in U(\mathbb{F}_{q^n})} K_{\mathcal{F},n}(x) \overline{K_{\mathcal{G},n}(x)}$$

<u>COROLLARY</u>: $\mathcal{F}^{\mathrm{geom}} = \mathcal{F}_{irr}^{N(\mathcal{G})}$ $\mathcal{F}_i$ irred geom

$\mathcal{G}^{\mathrm{geom}} \simeq \mathcal{G}_{irr}^{N(\mathcal{G})}$ $\mathcal{G}_i$ ——

$\exists \{\alpha_1, \dots, \alpha_{N(\mathcal{F})N(\mathcal{G})}\} = \mathbb{C}'$ SUCH THAT

$$C_n(\mathcal{F}, \mathcal{G}) = (\alpha_1^n + \dots + \alpha_{N(\mathcal{F})N(\mathcal{G})}^n)\,\delta_{\mathcal{F}_{irr} \simeq \mathcal{G}_{irr}} + O_{c(\mathcal{F})c(\mathcal{G})}(q^{-n/2})$$

DELIGNE THEOREM APPLIED TO $\mathcal{F} \otimes \mathcal{G}^*$

BECAUSE $K_{\mathcal{G}^*,n} = \overline{K_{\mathcal{G},n}}$

$$K_{\mathcal{F} \otimes \mathcal{G}^*} = K_{\mathcal{F}} \cdot \overline{K_{\mathcal{G}}}$$

IF YOU CAN EVALUATE THE SUMS $\rightsquigarrow$ YOU DECIDE IF $\mathcal{F}$ AND $\mathcal{G}$ HAVE THE SAME IRR COMPONENTS

<u>COROLLARY</u>: (KATZ DIOPHANTINE CRITERION FOR IRRED)

IF $\lim \mathcal{C}_n(\mathcal{F}, \mathcal{F}) = 1 \iff \mathcal{F}$ GEOM. IRRED.

EX: $\forall k$ Kl_k is $\overbrace{\text{GEOM IRRED}}$

$$Kl_k(y) = \widehat{Kl_{k-1}(x^{-1})}(y)$$

$$\frac{1}{q}\sum |Kl_k(y)|^2 = \frac{1}{q}\sum_{x\neq 0}|Kl_{k-1}(x)|^2$$

$$= \frac{1}{q}\sum_{x\neq 0}|Kl_{k-2}(x)|^2 =$$

$$= \frac{1}{q}\sum_{x\neq 0}|e_q(x)|^2 = \frac{q-1}{q}$$

TRACE FUNCTIONS OVER SHORT INTERVALS

$$\sum_{\substack{n\leq X\\ X<q}} K_F(n)\, V\left(\frac{n}{X}\right) = \frac{X}{q^{1/2}}\sum \widehat{K_F}(n)\,\hat{V}\left(\frac{nX}{q}\right)$$

$$\ll q^{\frac{1}{2}}\, \|\widehat{K_F}\|_\infty$$

$$\widehat{K_F}(y) = C(F, \mathcal{L}_{e_q(yx)})\cdot q^{1/2}$$

IF $F_{irr} \neq \mathcal{L}_\psi$ $\forall \psi$ ADDITIVE

$$\Rightarrow \sum_n K_F(n)\, V\left(\frac{n}{X}\right) \ll_{C(F)} q^{1/2}$$

$$?\ M(F) = \max_{X\leq q}\Big|\sum_{n\leq X} K(F)\Big| \leq q^{1/2}\log q$$

BILINEAR SUMS OF TRACE FUNCTIONS $\quad M, N < q$

$$\Big|\sum_{m\leq M}\sum_{n\leq N}\alpha_m\beta_n K(mn)\Big|^2 \underset{CS}{\leq}$$

$$\|B\|^2\sum_{1\leq m,m'\leq M}\alpha_m\overline{\alpha_{m'}}\sum_{n\leq N}K(mn)\overline{K(m'n)}$$

$n \to K(mn)\overline{K(mn)}$ IS THE TRACE FUNCTION OF

$[\times m]\mathcal{F} \otimes [\times m']\mathcal{F}^*$ DOES IT CONTAIN SOME $\mathcal{L}_\psi$

$$\frac{1}{q^{1/2}} \sum K(mn)\overline{K(m'n)}\, e_q(\gamma n) \underset{\substack{\uparrow \\ \text{PARSEVAL}}}{=}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

$$\frac{1}{q^{1/2}} \sum_{z \in \mathbb{F}_q} \hat{K}(\gamma z)\hat{K}(z) \qquad \gamma = \begin{pmatrix} m'/m & -\gamma/m \\ 0 & 1 \end{pmatrix}$$

THEOREM (DELIGNE, LAUMON, KATZ)

IF K IS A TRACE FUNCTION OF $\mathcal{F}$ NOT CONTAINING ANY $\mathcal{L}_\psi$ THEN

$\hat{K}$ " ——— " ——— $\hat{\mathcal{F}}$ " ———

$C(\hat{\mathcal{F}}) \ll C(\mathcal{F})^{10}$

" ———

PARCEVAL + KATZ CRITERION $\Rightarrow$

if $\mathcal{F}$ is GEOM IRR $\Longleftrightarrow$ $\hat{\mathcal{F}}$ is GEOM IRRED

$\hat{K}(\gamma z)$ IS THE TRACE FUNCTION OF $[\gamma]^*\hat{\mathcal{F}}$

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1$$
$$z \mapsto \gamma z$$

FOR WHICH γ's $\hat{\mathcal{F}} \simeq_{geom} [\gamma]^*\hat{\mathcal{F}}$

<u>THM:</u> IF $\mathcal{F}$ DOES NOT CONTAIN ANY $\mathcal{L}_\psi \otimes \mathcal{L}_\chi$

$\Rightarrow$ the set of γ's ABOVE IS OF SIZE $O_{C(\mathcal{F})}(1)$

COROLLARY: $\sum \alpha_m \beta_n K(mn) \underset{C(\mathcal{F})}{\leq} \|\alpha\| \cdot \|\beta\| (MN)^{\frac{1}{2}} \times \left(\frac{1}{M} + \frac{q^{1/2} \log q}{N} \right)^{1/2}$

$\mathcal{F} \not\simeq \mathcal{L}_\psi \otimes \mathcal{L}_\chi$

DEFINITION: THE GROUP of GEOM AUTOMORPHISM OF A SHEAF $\mathcal{F}$ $\mathrm{Aut}_{\mathcal{F}}(\overline{\mathbb{F}}_q) = \left\{ \begin{array}{l} \gamma \in PGL_2(\overline{\mathbb{F}}_q) \text{ SUCH THAT} \\ [\gamma]^* \mathcal{F} \simeq_{geom} \mathcal{F} \end{array} \right\}$

THEOREM: (FKM) $q \geq 7$ $\mathcal{F}$ GEOM. IRRED.

ONE OF THE FOLLOWING HOLDS

- $C(\mathcal{F}) > q$
- $q \nmid |\mathrm{Aut}_{\mathcal{F}}(\overline{\mathbb{F}}_q)|$ AND EITHER $|\mathrm{Aut}| \leq 60$

OR is A SUBGROUP OF THE NORMALIZ. OF A MAX. TORUS OF PGL_2.

- $q \mid |\mathrm{Aut}| \Rightarrow \mathcal{F} = G^* \mathcal{L}_\psi$ $\quad G \in PGL_2(\overline{\mathbb{F}}_q)$

$K_{\mathcal{F}}(x) = \psi(Gx)$

THEOREM: K A TRACE FUNCTION / ISOTYPIC SHEAF $\mathcal{F}$

$\mathcal{F} \not\supseteq \mathcal{L}_\psi \otimes \mathcal{L}_\chi$

$$\sum_{p \leq q} K(p) = O_{C(\mathcal{F})}\left(q^{1 - \frac{1}{48} + \varepsilon}\right)$$

RETURN TO d_3: $\quad LMN = q$

$$\sum_{\substack{m \sim M \\ n \sim N \\ l \sim L}} Kl_3(lmn, q) \ll q^{1-\eta} \qquad \eta > 0$$

IF M OR N OR $L \geq q^{\frac{1}{2} + \delta}$ WE CAN USE THE BILINEAR SUM BOUND

CAN ALSO SP 2 OF THE 3 VARIABLE TO FORM ONE $\geq q^{\frac{1}{2}+\delta}$

REMAINS IS THE CASE

$$L = 1 \qquad M \sim q^{\frac{1}{2}} \qquad N \sim q^{\frac{1}{2}}$$

$$\sum_{m,n \sim q^{1/2}} Kl_2(mn,q) \Longleftrightarrow \sum_{n \sim q} \lambda_g(n) Kl_3(n,q)$$

$\uparrow$

$\lambda_E(n)$ $\qquad \ll q^{1-\eta}$

THEOREM:

$$\sum_{m \sim q^{1/2}} \sum_{n \sim q^{1/2}} \alpha_m \beta_n Kl_R(mn,q)$$

$$\ll \|\alpha\| \cdot \|\beta\| \cdot q^{1-\eta+\varepsilon} \qquad \eta = \frac{1}{64} + \varepsilon$$

THEOREM: (ZACHARIAS)

$\chi_1, \chi_2\ (q)$

$\exists >$ PROPORTION OF $\chi(q)$ SUCH THAT

$$L(\chi, \tfrac{1}{q})\, L(\chi\chi_1, \tfrac{1}{q})$$

$$L(\chi\chi_2, \tfrac{1}{q}) \neq 0$$