TRACE FUNCTIONS AND SPECIAL FUNCTIONS

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Q: WHAT IS A SHEAF?

A: IT IS LIKE THE ESEMBLE OF

Q: WHICH

A: USE exp sum exp $\int$

$$Kl_2(x,a;q) = \sum_{x \in \mathbb{F}_q^*} e\left(\frac{x+ax^{-1}}{q}\right)$$

$$F(a) = \int_0^\infty e^{(x+ax^{-1})} \frac{dx}{x}$$

$$\parallel$$

$$J_0(\sqrt{a})$$

KLOOSTERMAN SHEAF

$$\frac{a\,d^2f}{da^2} + \frac{dF}{da} = 2\pi i F$$

① Ranks / sharp constants

② Monodromy / Equidistribution

③ Asymptotics / local monodromy (LOCAL FT)

1) RANKS / Sharp CONSTANTS

EX: 1.1) $|Kl_2(a|q)| \leq 2\sqrt{q}$

2 = RANK OF KLOOSTERMAN SHEAF

2 = ORDER OF THE DIFF. EQ.

2 = DIM. OF SPACE OF SOLUTIONS

PRINCIPLE:

SHARP CONSTANT IN WEIL BOUND = rank of sheaf

$\parallel$

= dim of space of solutions

ORDER OF DIFF. EQ.

CAN OFTEN BE COMPUTED BY STATIONARY PHASE

$$\int_0^\infty e(x+ax^{-1})\frac{dx}{x} = \int_0^\infty e(\sqrt{a}(x+x^{-1}))\frac{dx}{x}$$

$$\approx \frac{e(2\sqrt{a})}{\sqrt{\sqrt{a}}} + \frac{e(-2\sqrt{a})}{\sqrt{\sqrt{a}}}$$

$\frac{d}{dx}(x+x^{-1})=0$

$1-\frac{1}{x^2}=0$

$x=\pm 1$

EX. 1.2)

$$\int_{-\infty}^{\infty} e(f(x))dx$$

$$\left|\sum_{x\in\mathbb{F}_q} e\left(\frac{f(x)}{q}\right)\right| \leq (d-1)\sqrt{q}$$

f POLY OF DEG $d \neq 0$

dominate $f \to \infty$ by CRITICAL POINTS OF f

$d-1$ (ROOTS OF $\frac{df}{dx}$)

EX: 1.3

$$\left|\sum_{x_1,\dots,x_{k-1}\in\mathbb{F}_q} e\left(\frac{x_1+\dots+x_{k-1}+\frac{a}{x_1\cdots x_{q-1}}}{q}\right)\right| \leq k\,q^{\frac{k-1}{2}}$$

$x_1+\dots+x_{k-1}+\frac{1}{x_1\cdots x_{k-1}}$ HAS k CRITICAL POINTS

(2) MONODROMY / EQUIDISTRIBUTION

THM (KATZ-DELIGNE) VIEW $\frac{Kl_2(a;q)}{\sqrt{q}}$ OR A RANDOM

VARIABLE

a UNIFORMLY RANDOM IN $\mathbb{F}_q$

AS $q \to \infty$, THE DISTRIBUTION CONVERGES TO

IF WE DO THE SAME FOR $\frac{Kl_k(a!q)}{q^{k-1/2}}$ THE LIMIT IS

(k even) tr of a HAAR-RANDOM $g \in USp_k$

(k ODD) tr of a Haar-RANDOM ELEMENT SU_k.

THEOREM: (DELIGNE-KATZ)

FOR A GENERAL TRACE FUNCTION OF A REP

$\rho : \underline{Gal(\mathbb{F}_q(t)) \to GL_2}$ THE DISTRIBUTION IS CONTROLLED

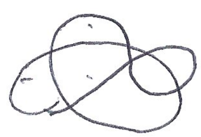
BY $\rho(Gal(\overline{\mathbb{F}}_q(t)))$ MONODROMY GROUP

PRINCIPLE: THE MONODROMY GROUP OF THE SHEAF IN THE MONODROMY OF THE DIFF. EQ AS LONG AS THE DIFF. EQ HAS REGULAR SINGULARITIES $f(z) \sim z^s$

MONODROMY OF DIFF EQ

GIVEN AN ODE ON $\mathbb{C}$ WITH SINGULARITIES $a_1 \dots a_n$

$x \in \mathbb{C} - \{a_1 \dots a_n\}$ V VECTOR SPACE OF SOLUTIONS AT X



GIVEN A PATH IN $X \to X$ IN $\mathbb{C} - \{a_1 \dots a_n\}$ WE CAN ANALYTICALLY CONTINUE ALONG THE PATH TO GET A MAP $V \to V$

$\rho : \underline{\pi_1(\mathbb{C} - \{a_1 \dots a_n\})} \to GL(V)$

$Im \rho$ = MONODROMY GROUP

EX: 2.1

$\chi(a)$ - order d

$r\chi$ - character sheaf   equidistributed in μ_d

$f(z) = z^{c/d}$ $(c,d)=1$

$$\frac{df}{dz} = \frac{c}{d}\frac{f}{z}$$

 z

$f \to e\left(\frac{c}{d}\right) f$

$\mathbb{Z} \to GL_1$

$n \mapsto e(nc/d)$

MONODROMY GROUP $= \mu_d$

EX. 2.2 χ_1, χ_2, χ_3 SPECIAL FUNCTIONS

$$\sum_{x \in \mathbb{F}_q \setminus \{0,1,t^{-1}\}} \chi_1(x)\chi_2(1-x)\chi_3(1-tx) = \text{tr (hypergeometric sheaf)}$$

$${}_2F_1(a,b,c,t) \quad \int_0^1 z^a (1-z)^b (1-tz)^c \, dz = \text{hypergeometric function}$$

SINGULARITIES AT 0

EX: 2.3 $e\left(\frac{a}{q}\right)$ $e(z)$ - trivial monodromy

non trivial distribution

L_ψ has monodromy μ_q

When the diff eq. has irregular singularities, we use differential Galois group = suitable automorphisms of solutions

Kl_R has irregular singularities at ∞

3. $J(a) \approx \dfrac{e(2\sqrt{a})}{\sqrt{\sqrt{a}}} + \dfrac{e(-2\sqrt{a})}{\sqrt{\sqrt{a}}} = \sum_{\substack{x \mid \\ x^2 = a}} \dfrac{e(2x)}{\sqrt{x}}$

$Kl_2(a;q)$ $\qquad\qquad$ $\sum_{x \mid x^2 = a} e\left(\frac{2x}{q}\right)\left(\frac{x}{q}\right)$

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KLOOSTERMAN SHEAF F $\qquad\qquad$ SHEAF G

$\rho_G = \mathrm{Ind}_{\mathbb{F}_q(\alpha)} L_\psi(2a \ldots)$

ρ_F are isomorphic of the restriction to $\mathrm{Gal}(\overline{\mathbb{F}_q}((t^{-1})))$

EXPANSION INTEGRALLS $\qquad$ LOCALLY LOOK LIKE

z^s $\qquad$ L_χ

$\log z$ $\qquad$ UNIPOTENT REP

$e^{f(z)}$ $\qquad$ $L_\psi(f(z))$

$p >> 1$ $\qquad$ (GROTHENDIECK)

$\log z \to \log z + 2\pi i$

$2\pi i \to 2\pi i$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \log z \\ 2\pi i \end{pmatrix}$

$$\frac{\sum_x e\left(\frac{t f(x)}{q}\right)}{\sqrt{q}}$$

is DISTRIBUTED ACCORDING TO A GROUP G $\quad SU_{\delta - p} \leq G \leq U_{d-1}$

AS LONG AS f HAS

1) SIMPLE CRITICAL POINTS

2) $a_1 \ldots a_{d-1}$ CRITICAL POINTS

$f(a_1) \ldots f(a_{d-1})$ CRITICAL VALUES

SATISFY NONTRIVIAL RELATIONS

$f(a_i) = f(a_j)$ $\qquad$ $\mathrm{Im}\,\rho = M$

$f(a_i) + f(a_j) = f(a_k) + f(a_l)$