THE KUZNETSOV FORMULA, KLOOSTERMANIA AND APPLICATIONS

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GEOMETRIC SIDES:

PETERSSON: $\displaystyle\sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$

KUZNETSOV: $\displaystyle\sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c} \int_{-\infty}^{\infty} J_{-2it}(x)\, \frac{h(t)}{\cosh \pi t}\, dt$

CAN CHECK: $J_{2it}(x) - J_{-2it}(x) \perp J_{k-1}(x) \in L^2(\mathbb{R}_{>0}, \frac{dx}{x})$

LET $f \in C^2(\mathbb{R}_{>0})$ SUCH THAT $f(0)=0$ $f(x) \ll (1+x)^{-\alpha}$

$$\mathcal{M}_f(t) = \frac{\pi i}{\sinh \pi t} \int_0^\infty \left(J_{2it}(x) - J_{-2it}(x)\right) f(x)\, \frac{dx}{x} \qquad \begin{matrix}\alpha > 2+\delta \\ a=0,1,2\end{matrix}$$

$$\mathcal{N}_f(k) = \frac{4(k-1)!}{(4\pi i)^k} \int_0^\infty J_{k-1}(x)\, f(x)\, \frac{dx}{x} \qquad \Gamma = \Gamma_0(q)$$

$\{u_j\}$ COMPLETE O.N. SET FOR $L_0^2(\Gamma\backslash\, d\mu)$

$\rho_j(n)$ FOURIER COEFFS OF u_j

$\tau_{\mathfrak{a}}(n,t)$ " " $E_{\mathfrak{a}}(z, \frac{1}{2}+it)$

$a_g(n)$ " " $g \in \mathcal{B}_k(q, \text{triv})$

<u>THM</u> (BRUGGEMAN-KUZNETSOV-DESHOUILLERS-IWANIEC) $mn>0$

$$\sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c} f\left(\frac{4\pi\sqrt{mn}}{c}\right) = \sum_{j=1}^{\infty} \mathcal{M}_f(t_j)\, \overline{\rho_j(m)}\, \rho_j(n)$$

$$+ \sum_{\mathfrak{a} \text{ CUSP OF } \Gamma_0(q)} \frac{1}{4\pi} \int_{\mathbb{R}} \mathcal{M}_f(t)\, \overline{\tau_{\mathfrak{a}}(m,t)}\, \tau_{\mathfrak{a}}(n,t)\, dt \; +$$

$$+ \sum_{\substack{k \geq 2 \\ k\,\text{even}}} N_f(k) \sum_{g \in B_k(a,\,\text{triv})} \overline{a_g(m)}\, a_g(n)$$

REMARKS: ① CAN ALSO DO $mn<0$ WITH DIFFERENT INTEGRAL TRANSFORMS

② THE ABOVE FORMULA IS USING POINCARÉ SERIES @ ∞

FOURIER EXPANSION @ ∞

KLOOSTERMANIA

$$\begin{tikzcd} \text{NUMBER THEORY} \arrow[r, bend left] & \text{AUT. FORMS} \arrow[l, bend left] \end{tikzcd}$$

$u_0, u_1, u_2 \ldots$ MASS FORMS $\Delta u_j = \lambda_j u_j$

$u_0 =$ CONST. FUNCTION

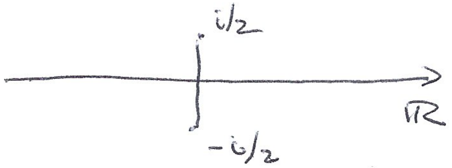
$\lambda_0 = 0$

$\lambda_j = s_j(1-s_j) = \frac{1}{4} + t_j^2$

$\lambda_j \geq 0$

$s_j = \frac{1}{2} + i t_j$

$\lambda_j \geq 0 \Rightarrow t_j \in$



SELBERG EIGENVALUE CONJECTURE

$$\lambda_1 \geq 1/4 \iff \mathrm{Re}(s_j) = 1/2$$
$$\iff t_j \in \mathbb{R}$$

AS $\Gamma_0(q)$ VARIES

SMALLEST EVAL MASS FORM FOR $SL_2\mathbb{Z}$, $\lambda_1 \approx 91\ldots$

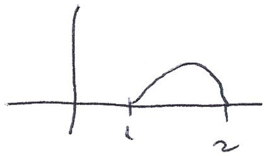
WORLD RECORD (KIM - SHAHIDI - SARNAK):

$$\lambda_1 > \frac{975}{4096}$$

SELBERG : $\lambda_1 \geq 3/16$

PROOF : TAKE $m=n$.

LET φ :



LET $f(x) = \varphi({}^{Y}\!/_{x})$ $Y \geq 10$

LHS

$$\sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c}\,\varphi\left(\frac{4\pi\, {}^{Y}\!/_{n}}{c}\right)$$

$$\underset{\text{Weil}}{\leq} \sum_{\substack{c\equiv 0(q)\\ c\leq 8\pi nY}} \frac{(n,c)^{1/2}\, d(c)}{c^{1/2}} \ll_q (nY)^{1/2} \log nY$$

WRITE $\tilde{\varphi}(s) = \int_0^\infty \varphi(x)\, x^s \,\frac{dx}{x}$ $t\in\mathbb{R}$

THEN $M_{\varphi(\cdot Y)}(t) \ll \frac{|\tilde{\varphi}(2it) - \tilde{\varphi}(-2it)|}{e^{\pi|t|}}$ (uniformly in Y)

$N_{\varphi(Y)}(k) \ll \frac{1}{(4\pi Y^k)}$

THE WHOLE SPECTRAL SIDE CONVERGES TO $O_{n,q}(1)$ AS $Y\to\infty$ EXCEPT IF u_j is SUCH THAT $t_j \notin \mathbb{R}$.

$$\sum_{\substack{u_j\\ \text{EXCEPTIONAL}}} M_{\varphi(\cdot Y)}(t_j)\,|\rho_j(n)|^2 \ll_{n,q} Y^{\frac{1}{2}} \log Y$$

SUPPOSE NOW $it_j \in \mathbb{R}$
$0 < it_j \leq 1/2$

$$M_{\varphi(\cdot Y)}(t) = \int_0^\infty \left(J_{2it}(x) - \underset{\substack{\uparrow\\ \text{DOMINATES}}}{J_{-2it}}(x)\right)\varphi(xY)\,\frac{dx}{x}$$

$$J_{-2it}(x) \underset{x\to 0}{\sim} \frac{1}{\Gamma(1-2it)} \left(\frac{x}{2}\right)^{-2it}$$

$$\Rightarrow \quad M_{\varphi(\cdot Y)}(t) \underset{Y\to\infty}{\sim} c\, Y^{2it}$$

$$\Rightarrow \quad \sum_{\substack{u_j \\ \text{exceptional}}} Y^{2it} |\rho_j(n)|^2 \ll_{n,q} Y^{1/2} \log Y$$

$Y \to \infty$

$\Rightarrow \quad it \leq 1/4$

$\Rightarrow \quad \lambda_1 \geq 3/16$

$$\sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c}\, \psi\left(\frac{c}{Y}\right) \ll_{m,n} Y^{\theta+\varepsilon}$$

$\psi \in C^\infty(\mathbb{R}_{>0})$

non-oscillatory

$\Rightarrow \quad |\operatorname{Im} t_1| \leq \theta/2$ ie

$$\lambda_1 \geq \frac{1}{4} - \frac{\theta^2}{4}$$

SAME ARGUMENT $\Rightarrow \displaystyle\sum_{c\equiv 0(q)} \frac{S(m,n,c)}{c}\, \psi\left(\frac{c}{Y}\right)$

$$\ll X^{\sqrt{\max(0,1-4\lambda_1)}+\varepsilon}$$

—— x ——

FOUVRY - IWANIEC, FOUVRY, BOMBIERI - FRIELANDER, IWANIEC

SAY $\lambda : \mathbb{N} \to \mathbb{C}$ IS OF LEVEL D if $\lambda_d = 0 \ \forall d > D$

AND EXPONENT K $\quad |\lambda_d| \leq \tau_K(d)$

SAY

λ IS WELL-FACTORABLE IF FOR ALL $D_1 D_2 = D$ THERE EXIST μ, ν OF LEVELS D_1, D_2 SUCH THAT $\lambda = \mu * \nu$

THEOREM (FI, F, BFI) LET $a \neq 0$ $\varepsilon > 0$ $x \geq 2$ $A > 0$

FOR ANY WELL-FACTORABLE λ OF LEVEL $Q = X^{4/7-\varepsilon}$

$$\sum_{(q,a)=1} \lambda(q) \Bigg(\sum_{\substack{p \equiv a(q) \\ p \leq x}} 1 - \frac{1}{\varphi(q)} \sum_{p \leq x} 1 \Bigg) \ll \frac{X}{(\log X)^A}$$

$\ll$ DEPENDS ON $\underset{\substack{\uparrow \\ \text{SAD}}}{a}, \varepsilon, A, K$

WE WOULD ESPECIALLY LIKE UNIFORMITY IN a

MAIN INPUT :

THM (DESHOUILLIER - IWANIEC):

LET $C, M, N > 0$ $g(m,n,c)$ BE A NON-OSCILLATORY FCN SUPPORTED ON $[M, 2M] \times [N, 2N] \times [C, 2C]$

a_m, b_n ANY SEQUENCES

THEN
$$\sum_c \sum_m \sum_n a_m b_n \overbrace{\vphantom{S}\ }^{g(m,n,c)} S(m, \pm n, c)$$
$$\ll C^{1+\varepsilon} \sqrt{MN} \, \|a\|_2 \, \|b\|_2$$

MORE GENERALLY:

USING CUSPS OTHER THAN ∞

LET $(a, r) = 1$.

$$\sum_{(c,r)=1} \sum_m \sum_n a_m b_n \, g(m,n,c) \, S(m\bar{r}, In, cq) \ll \text{strong bound}$$

Study: $$\sum_{(a,q)=1} \lambda(q) \Bigg(\sum_{\substack{n \equiv a(q) \\ n \leq x}} \Lambda(n) - \frac{1}{\varphi(q)} \sum_{(n,q)=1} \Lambda(n) \Bigg)$$

LEMMA (HEATH-BROWN IDENTITY)

LET $Z \geqslant 2$ $\quad J \in \mathbb{N}$ $\quad 2Z^J > n$

$$\Lambda(n) = \sum_{j=1}^{J} (-1)^{i} \binom{J}{j} \sum_{m_1 \cdots m_j < Z} \mu(m_1) \cdots \mu(m_j) \sum_{m_1 \cdots m_j n_1 \cdots n_j} \log n_i$$

USE HEATH-BROWN IDENTITY WITH $J=4$

LET $M_1, \ldots, M_j;\ N_1, \ldots, N_j$

NEED TO HANDLE:

$$\mathcal{E}(M_1 M_2 \ldots M_j ; N_1 \ldots N_j) = \sum_{(q,a)} \lambda(a) \sum_{\substack{m_i \sim M_i \\ n_i \sim N_i \\ m_1 \cdots m_j n_1 \cdots n_j = n}} \mu(m_1) \cdots \mu(m_j)$$

(+ an admissible error)

$$\times \left(\mathbb{1}_{n \equiv a(q)} - \frac{1}{\varphi(q)} \mathbb{1}_{\substack{(n,q) \\ =1}} \right)$$

ARRANGE $M_1 \ldots M_j$, $N_1 \ldots N_j$ INTO TWO SETS TO GET BILINEAR FORMS

BILINEAR FORMS:

LET $(\alpha_q)(\beta_r)(\gamma_m)(\delta_n)$ SEQUENCES BOUNDED BY τ_k SUCH THAT δ_n IS WELL-DISTRIBUTED IN RESIDUE CLASSES

ie
$$\sum_{\substack{n \equiv a(q) \\ n \sim N}} \delta_n - \frac{1}{\varphi(q)} \sum_{\substack{(n,q) \\ n \sim N}} \delta_n \ll N / (\log N)^A$$

$$\mathcal{E}(M, N, Q, R) \overset{\text{def}}{=} \sum_{q \leq Q} \alpha_q \sum_{r \leq R} \beta_r \left(\sum_{m \sim M} \gamma_m \sum_{\substack{n \sim N \\ mn \equiv a(q)}} \delta - \frac{1}{\varphi(a)} \sum_{\substack{(n,q)=1 \\ n \sim N}} \delta_n \right)$$

$MN \leq X$

GOAL: FOR $MN \leq X$ $\quad Q \leq N/X^{\varepsilon}$ $\quad R \leq M^{1/2}$

$$|\mathcal{E}(M, N, Q, R)| \ll X/(\log X)^A$$

CRUCIAL IN COMBINATORIAL DECOMP. OF $M_1 \ldots M_j$, $N_1 \ldots N_j$ THAT Q, R DEPEND ON M, N

SO, HERE WE NEED λ IS WELL-FACTORABLE.

LET $\gamma(m)$, $\alpha(q)$ SMOOTH APPROX. TO INTERVALS

$$|\mathcal{E}(\ldots)|^2 \ll QM(\log X)^{2k^2} \mathcal{D}$$

$$\mathcal{D} = \sum_q \alpha(q) \sum_m \gamma(m) \Big(\sum_{r \sim R} \beta_r \sum_{\substack{mn \equiv a(q) \\ n \sim N}} \delta_n - \frac{1}{\varphi(q)} \sum_{r \sim R} \beta_r \sum_{\substack{(n,q)=1 \\ n \sim N}} \delta_n \Big)$$

SQUARE OUT: $W - 2V + U$

W IS HARDEST, 4 VARIABLES, r_1, n_1, r_2, n_2

$\Rightarrow$ KLOOSTERMAN SUMS mod $r_1 n_2$