L-FUNCTIONS, SPECTRAL SUMMATION FORMULAS, MOMENTS OF L-FUNCTIONS

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(I) L-FUNCTIONS

a_n INTERESTING SEQUENCE $\quad L(s) = \sum \frac{a_n}{n^s}$

EX: $a_n = 1$ $\quad \zeta(s) = \sum \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$

$$\zeta(1+it) \neq 0 \iff \pi(x) \sim \frac{x}{\log x}$$

EX: $a(n) = \chi(n)$

$\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}$

$L(s, \chi) = \sum \frac{\chi(n)}{n^s}$

EX: AUTOMORPHIC FORM

$T_n f = \lambda(n) f$

$L(s, f) = \sum \frac{\lambda(n)}{n^s}$

GOAL: study $L(s)$ in the critical strip

typical approach: study families of L-functions

(II) SPECTRAL Σ FORMULAS

EX 1: $\sum_\chi \overline{\chi(n)} \chi(m) = \delta_{n=m} \varphi(q)$

EX 2: (PETERSSON)

$\sum_f a_f(n) \overline{a_f(m)}$ = simple factors $(\delta_{n=m} + \dots)$

EX 3: (SELBERG TRACE FORMULA)

$$\sum_{\text{auto rep } \pi \text{ on } G} h(\pi) \rightsquigarrow \text{algebraic and geometric properties of } G$$

APPLICATIONS:
- WEYL'S LAW
- VERTICAL SATO-TATE
- MOMENTS OF L-FUNCTIONS

III SIEGEL MODULAR FORMS OF DEGREE 2

$$\mathbb{H}^{(2)} = \{X + iY \in \mathrm{Sym}_2(\mathbb{C}) \mid Y > 0\}$$

weight k SIEGEL MODULAR FORM

$$f(\underbrace{mZ}_{(AZ+B)(CZ+D)^{-1}}) = \det((CZ+D))^k f(Z)$$

FOR ALL $m = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z}) = \Gamma$

Dim $\sim k^3$

HECKE OPERATORS T_p $\qquad \Gamma \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} \Gamma = \bigcup \Gamma g_i$

FOURIER EXPANSION $\qquad f(Z) = \sum_{T \in \mathrm{Sym}^+(\mathbb{Z})} a(T) (\det T)^{\frac{k}{2} - \frac{3}{4}} e(\mathrm{tr}(TZ))$

WARNING: $a(T)$ HAS LITTLE TO DO WITH HECKE EIGENVALUES

LIFT: $\quad S_{2k-2} \longmapsto S_{k-\frac{1}{2}} \xmapsto{\text{Saito-Kurokawa}} S_k^{(2)}$

$$f = \sum a(n) e(nz) \in S_{2k-2}, \qquad g = \sum b(n) e(nz) \in S_{k-\frac{1}{2}}, \qquad F = \sum a(T) e(\mathrm{tr}\, TZ) \in S_k^{(2)}$$

$$a(T) \sim b(\det T)$$

BOCHERER'S CONJECTURE (80'S)

$$a(I)^2 \sim \frac{L(\tfrac12, F)\, L(\tfrac12, F \times \chi_{-4})}{L(1, F, Ad)}$$

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F NOT A LIFT

PETERSSON FORMULA (KITAOKA)

$$\sum_F a_F(T)\, a_F(Q) \sim diag + \sum_C \frac{K(Q,T,C)}{|\det C|^{3/2}} J(TC^{-1}QC^{-T})$$

$$K(Q,T,C) = \sum_{\begin{pmatrix} A & * \\ C & D \end{pmatrix} \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} e\left(tr(AC^{-1}Q + C^{-1}DT)\right)$$

L-FUNCTION: $\zeta_{Q(i)}(s + 1/2) \sum \frac{a_F(mI)}{m^s} = a_F(I)\, L(s, F)$

IV RESULTS (B -2016)

TH I: $\frac{1}{k^3} \sum_F a_F(I)^2\, L\left(\frac12, F \times \chi_{q_1}\right) L\left(\frac12, F \times \chi_{q_2}\right)$

$$= MT + O(k^{-\frac12 + \varepsilon})$$

TH II: (quadr nonvanishing)

FIX q_1, q_2. THEN $\exists$ A NON-LIFT F SUCH THAT

$L(\tfrac12, F)$, $L(\tfrac12, F \times \chi_{-4})$, $L(\tfrac12, F \times \chi_{q_1})$

$L(\tfrac12, F \times \chi_{q_2})$ SIMULTANEOUS NONZERO

TH III: [outlook]

similar results in the level aspect

(V) IDEA OF THE PROOF

- APPROXIMATE FUNCTIONAL EQUATION
  (BONUS: FEATURES ONLY DIAGONAL MATRICES)
- APPLY PETERSSON
- APPLY POISSON
- ANALYSE THE FOURIER OF THE MATRIX BESSEL FUNCTION

  ONLY $C = \begin{pmatrix} x & y \\ \mp y & \pm x \end{pmatrix} \in GO_2(\mathbb{Z})$ SURVIVE

- FOR THESE ANALYSE THE FOURIER TRANSF OF KLOOSTERMAN Σ $\rightarrow$ ARITHMETIC OF $\mathbb{Q}(i)$
- MATCHES EXTRA OFF Δ MAIN TERM WITH PART OF Δ TERM.