Configurations in sets big and small

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Connections for Women: Harmonic Analysis Program

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Plan

- Set-up

- History

- Avoidance

- Existence

- Abundance

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Structures in sets: some general questions

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What is big? What is small?

Quantification of size

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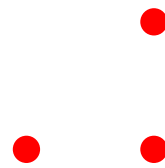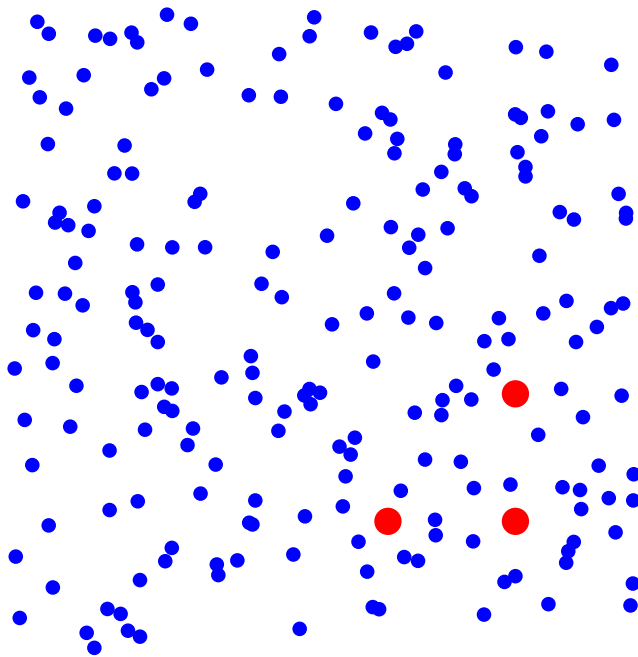
- dimension(s)

- density

What is a configuration?

- In principle, any prescribed set.

- Could be

  - **geometric**, such as specially arranged points on a line, vertices of an equilaterial triangle, or

  - **Algebro-analytic**, for example solutions of a polynomial equation.
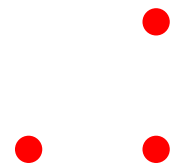
A set and a configuration
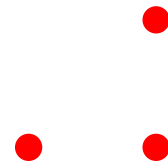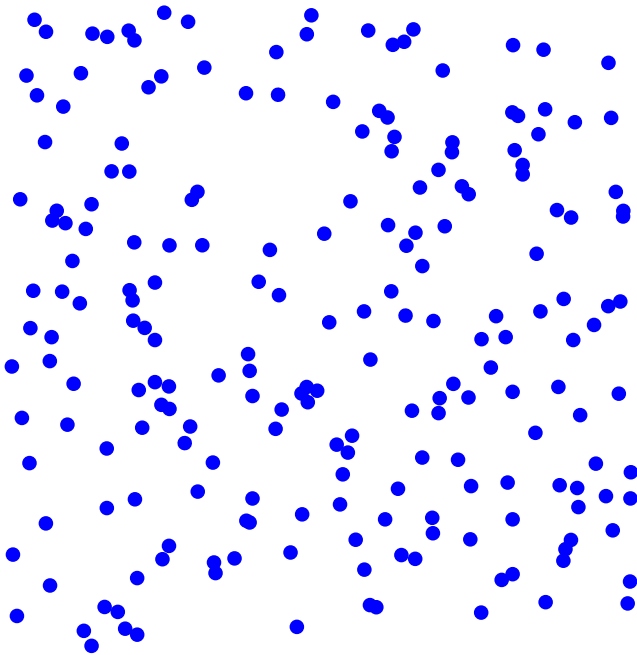
A congruent copy of the triangle

An affine copy of the triangle
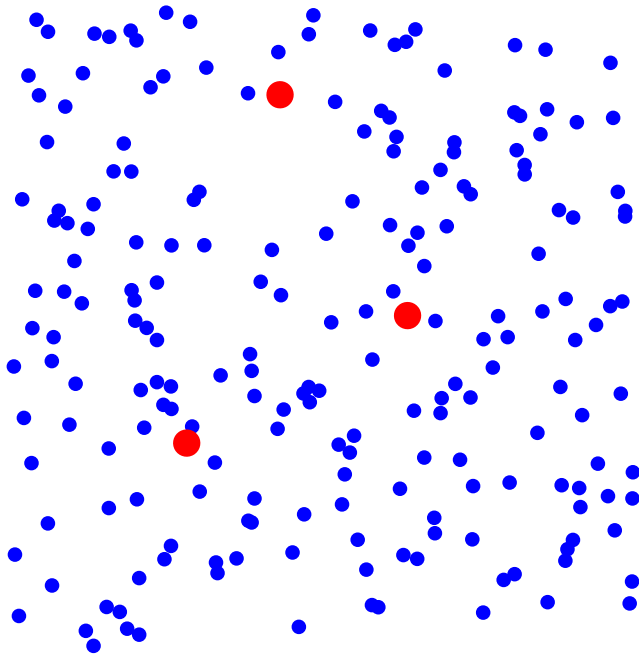
An example: Progressions in the integers

Theorem (Szemerédi 1975)

If $E \subseteq \mathbb{N}$ with positive asymptotic density, i.e.,

$$\limsup_{N \to \infty} \frac{\#(E \cap [1, N])}{N} = \delta > 0,$$

then E contains an arithmetic progression of length k, for any $k \geq 3$.

Example (ctd): Progressions in large sets of zero density

- (Salem and Spencer 1942, Behrend 1946) There are large sets
  $E_N \subseteq \{1, 2, \cdots, N\}$,

$$\#(E_N) > N^{1 - c/\sqrt{\log N}}$$

  that contain no three-term arithmetic progression.

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  that contain no three-term arithmetic progression.

- But there are other large sets of zero density that have long
  progressions!

  - random sets (Kohayakawa, Łuczak and Rödl 1996),

  - primes (Green and Tao 2008), ...

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Szemerédi-type problems in the continuum

A prototypical question

Given

- a class of subsets $\mathcal{E}$ in $\mathbb{R}^n$ whose members are "large", and

- a choice of geometric configurations $\mathcal{F}$

must every set $E \in \mathcal{E}$ contain one of the prescribed configurations $F \in \mathcal{F}$?

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For instance,

- given a fixed $F \subseteq \mathbb{R}$, is there a geometrically similar copy of F in every set of positive Lebesgue measure? Here $\mathcal{E} =$ sets of positive Lebesgue measure, $\mathcal{F} =$ similar copies of F.

- If yes, call F **universal**.

Finite sets are universal

Theorem (Steinhaus 1920)

Given

- any finite set $F \subset \mathbb{R}$, and

- any set $E \subseteq \mathbb{R}$ of positive Lebesgue measure,

there exists $x \in \mathbb{R}$ and $t \neq 0$ such that $x + tF \subseteq E$.

Steinhaus's theorem - a special case

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- Lebesgue density theorem $\implies$ almost every $x \in E$ is a density point. What this means is that $\exists x \in E$ such that

$$\lim_{r \to 0} \frac{|E \cap (x - r, x + r)|}{2r} = 1.$$

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- If E has no affine copy of F, then for every $t > 0$, either $x - t \notin E$ or $x + t \notin E$, so

$$|E \cap (x - r, x + r)| \leq \frac{2r}{2} = r \qquad \forall r,$$

contradicting Lebesgue density!

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> **Erdős similarity problem 1974**
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> Does there exist an infinite universal set?

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- An earlier question:

  "If $\{x_n\}$ is an infinite sequence $\to 0$, then for every $E \subseteq \mathbb{R}$, $|E| > 0$, does $\exists x \in \mathbb{R}$ such that $x + x_n \in E$ for all sufficiently large n?"

  answered in the negative by Borwein and Ditor 1978.

Erdős similarity problem : progress so far

Conjecture restated

Given any infinite set $F \subseteq \mathbb{R}$, there exists a set E of positive measure which does not contain any nontrivial affine copy of F.

- Conjecture verified for

    ▸ slowly decaying sequences $\{x_i\}$, where $x_{i+1}/x_i \to 1$ (Falconer 1984),

    ▸ $S_1 + S_2 + S_3$, where each S_j is infinite (Bourgain 1987),

    ▸ $\{2^{-n^\alpha}\} + \{2^{-n^\alpha}\}$ for $0 < \alpha < 2$ (Kolountzakis 1997).

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- Not known even for $\{2^{-n} : n \geq 1\}$.

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Now what?

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- Finding similar copies of infinite patterns in sets of positive Lebesgue measure seems to be hard, but ...

- Can one find other large Lebesgue-null sets that contain affine copies of all finite sets?

Dimension: an alternative notion of size

Given $E \subseteq \mathbb{R}^n$, recall

Definition

$$\dim_{\mathbb{H}}(E) := \sup \left\{ \alpha \in [0, n] \;\middle|\; \begin{array}{l} \exists \text{ a probability measure } \mu, \ \mathrm{supp}(\mu) \subseteq E, \\[1ex] \displaystyle \sup_{x \in \mathbb{R}^n} \sup_{\epsilon > 0} \frac{\mu(B(x, \epsilon))}{\epsilon^{\alpha}} < \infty \end{array} \right\}.$$

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- Agrees with the standard notion of dimension for curves, surfaces etc.

- Assigns a quantitative measure of size to less regular objects, such as fractals.

- The Cantor middle-third set has Hausdorff dimension $\log 2 / \log 3$.

Dimension of fractals: some examples

Sierpinski triangle

dimension $= \log_2 3$

Dimension of fractals: some examples

Graph of Brownian motion
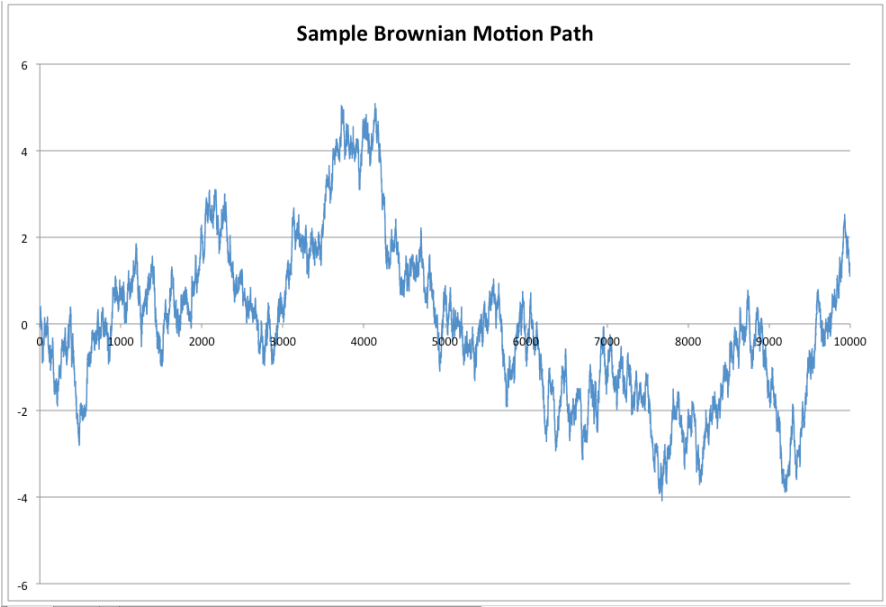
dimension = 3/2

Dimension: an alternative notion of size
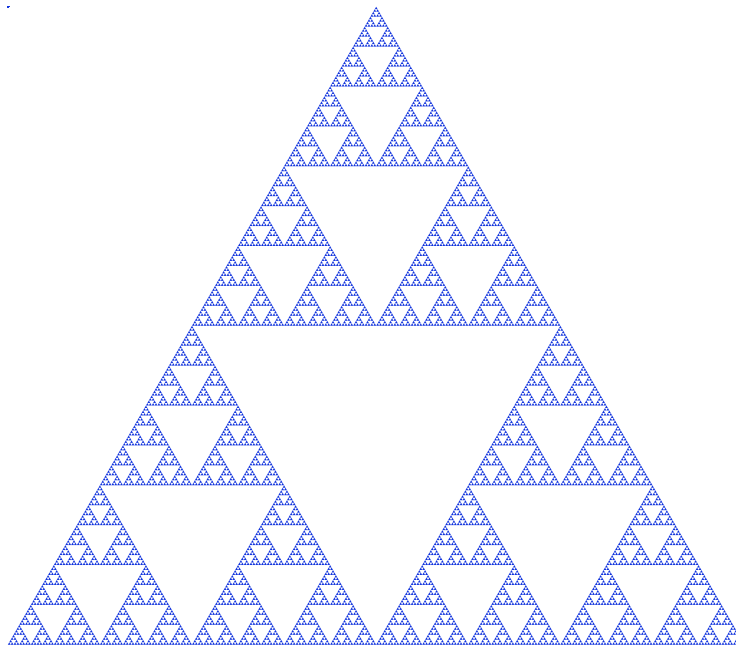
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- Every set of positive Lebesgue measure has full Hausdorff dimension.

- Converse is not true. There are many Lebesgue-null sets of full dimension.

Universality revisited

Revised questions

- Does there exist a Lebesgue-null subset of $\mathbb{R}$ of full Hausdorff dimension containing an affine copy of every finite configuration?

Universality revisited

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- Does there exist a Lebesgue-null subset of $\mathbb{R}$ of full Hausdorff dimension containing an affine copy of every finite configuration?

- If the answer to the above is yes, must every full-dimensional Lebesgue-null set have this property?

Yes! and No!

> **Theorem (Erdős and Kakutani 1957)**
>
> There exists a compact Lebesgue-null set in $\mathbb{R}$ of Hausdorff dimension 1 containing similar copies of all finite subsets.

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Theorem (Erdős and Kakutani 1957)

There exists a compact Lebesgue-null set in $\mathbb{R}$ of Hausdorff dimension 1 containing similar copies of all finite subsets.

Theorem (Keleti 1998, 2008)

- For a given distinct triple of points $\{x, y, z\}$, there exists a compact set in $\mathbb{R}$ with Hausdorff dimension 1 which does not contain any similar copy of $\{x, y, z\}$.

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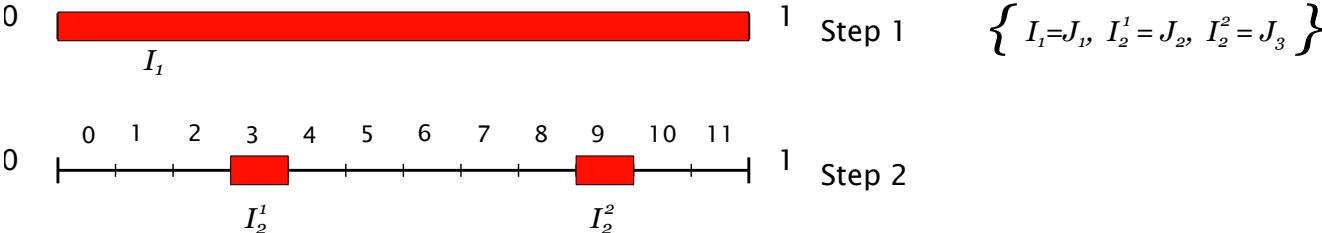- Given any countable $A \subset (1, \infty)$, there exists a compact set $E \subset \mathbb{R}$ with Hausdorff dimension 1 such that if $x < y < z$, $x, y, z \in E$ then $(z - x)/(z - y) \notin A$.

Keleti's example: Cantor construction with memory

0   I_1 1   Step 1 $\left\{ I_1 = J_1, \; I_2^1 = J_2, \; I_2^2 = J_3 \right\}$

0   1   2   3   4   5   6   7   8   9   10   11

0   I_2^1 I_2^2 1   Step 2

Keleti's example: Cantor construction with memory

$$\left\{ J_1, \, J_2, J_3, \, ..., J_{m\text{-}1}, \, ..., J_N = I_{m\text{-}1}^{(m\text{-}1)!} \right\}$$

Step 1

Step 2

Step m−1

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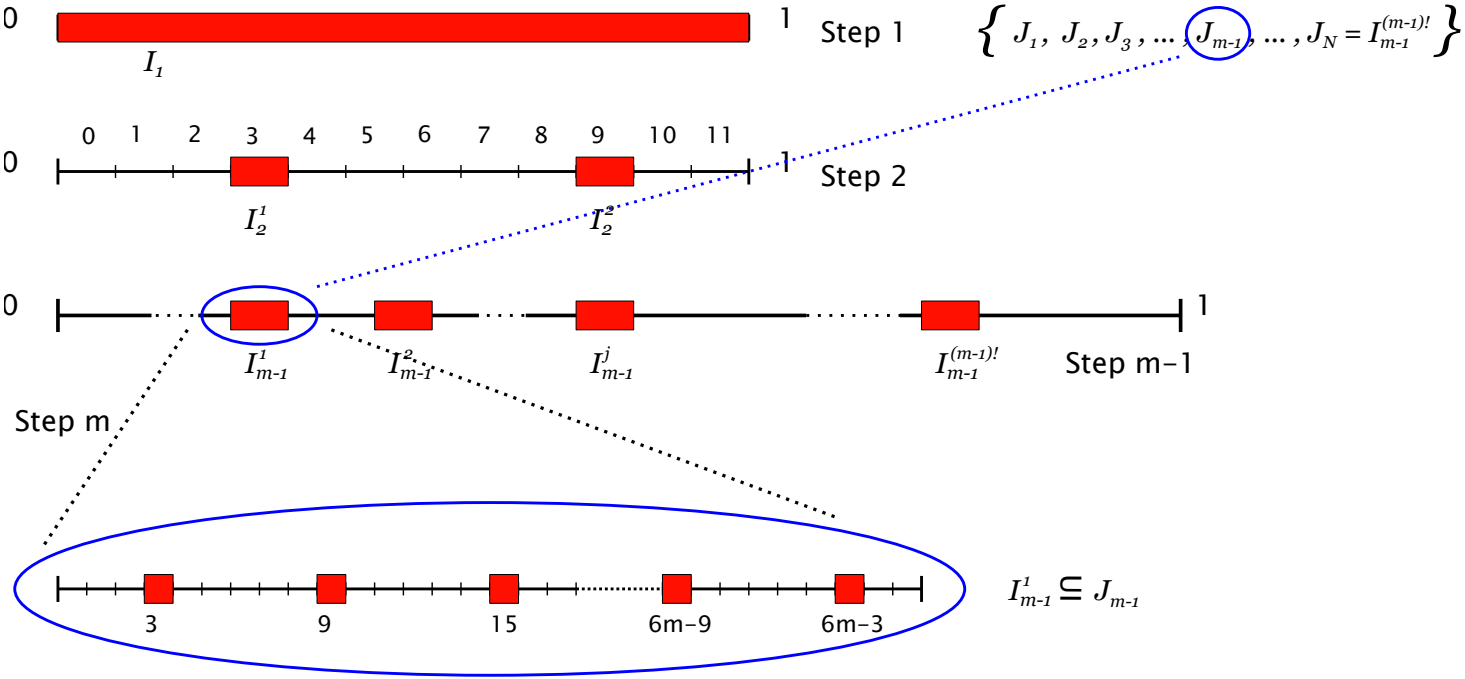
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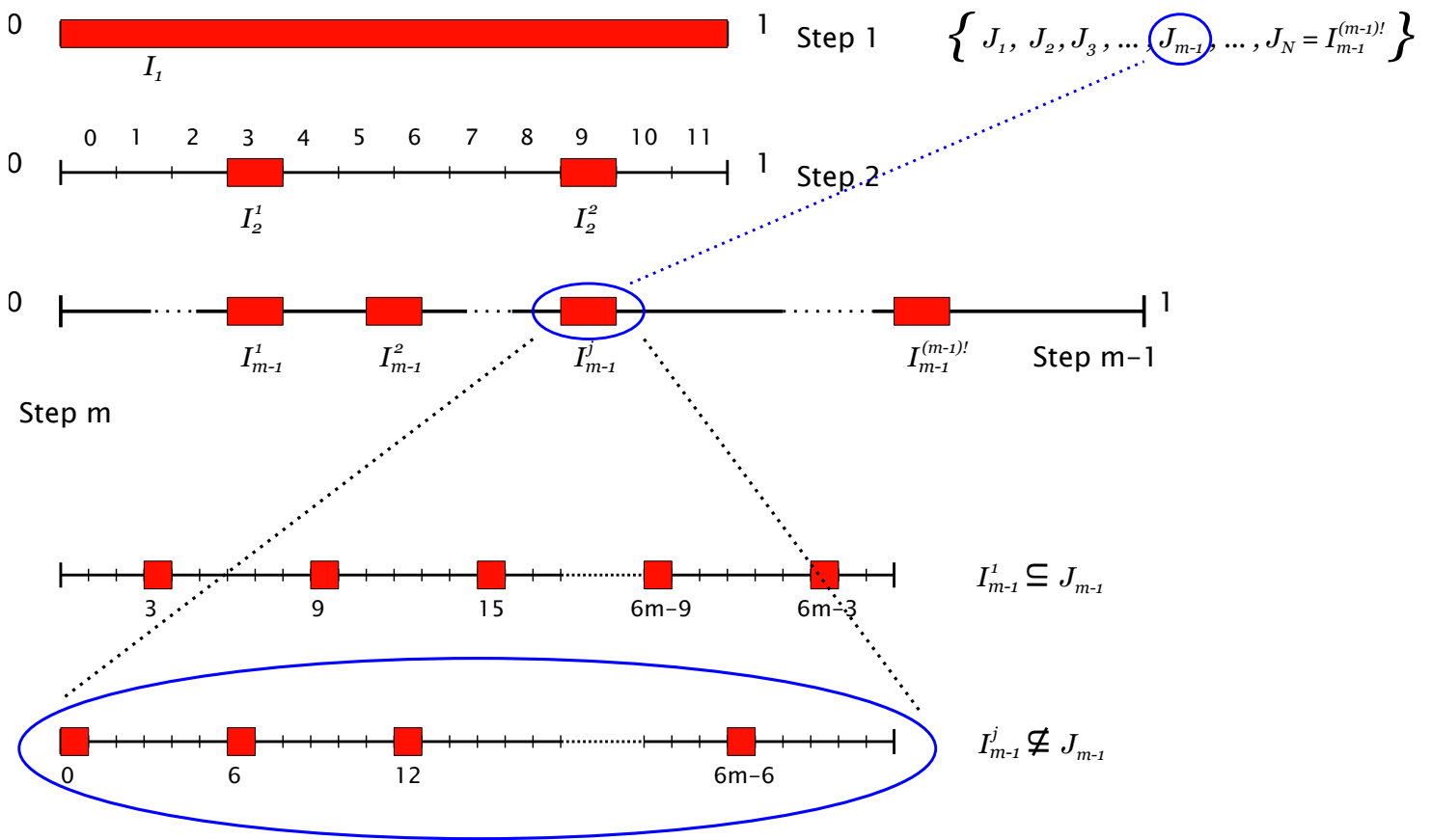
Avoidance of $x_1 < x_2 \leq x_3 < x_4$ with $x_2 - x_1 = x_4 - x_3$

Let $x_1 \in J_{m-1}$, $x_2, x_3, x_4 \notin J_{m-1}$.
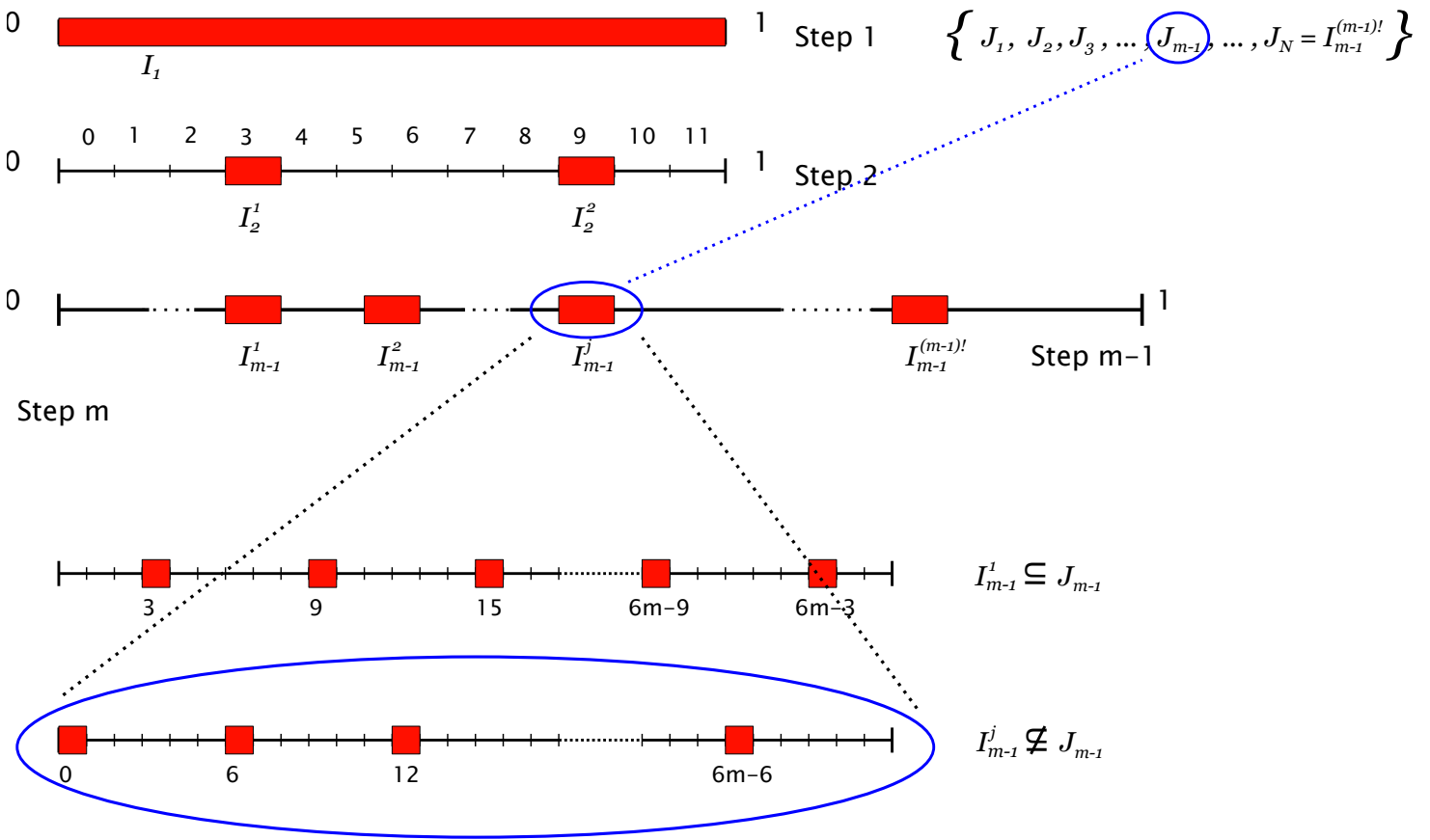
At step m, $x_1 \in$ an interval indexed by $6\mathbb{Z} + 3$, but x_2, x_3, x_4-index is $6\mathbb{Z}$!

Higher dimensional configurations: different points of view

Theorem (Maga 2010)

(a) For distinct $x, y, z \in \mathbb{R}^2$, there exists a compact set in $\mathbb{R}^2$ with Hausdorff dimension 2 not containing any similar copy of $\{x, y, z\}$.

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(a) If $E \subseteq \mathbb{R}^2$ is compact with $\dim_{\mathbb{H}}(E) = 2$, must E contain the vertices of an isosceles triangle?

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(a) If $E \subseteq \mathbb{R}^2$ is compact with $\dim_{\mathbb{H}}(E) = 2$, must E contain the vertices of an isosceles triangle?

(b) Given a set $E \subset \mathbb{R}^n$, how large can $\dim_{\mathbb{H}}(E)$ be if E does not contain a triple of points forming a particular angle θ?

Avoidance: angles, functional zeros

Angles in sets

Say $\angle\theta \in E \subset \mathbb{R}^n$ if there exist distinct points $x, y, z \in E$ such that the angle between the vectors $y - x$ and $z - x$ is θ. Define

$$C(n, \theta) := \sup\left\{s : \exists E \subseteq \mathbb{R}^n \text{ compact with } \dim_{\mathbb{H}}(E) = s, \ \angle\theta \notin E\right\}$$

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- For $\theta = 0, \pi$, $C(n, \theta) = n - 1$.

- For $\theta = \frac{\pi}{2}$, $n/2 \leq C(n, \theta) \leq \lfloor (n + 1)/2 \rfloor$.

- Sets of dimension $1/d$ avoiding zeros of a multivariate polynomial of degree d with rational coefficients (Mathé 2012)

- Large sets of special Fourier-analytic structure avoiding *all* k-variate rational linear relations (Körner 2009)

- Generalizations involving non-polynomials (Fraser-P 2016)

Avoidance of functional zeros

Theorem (Fraser-P 2016)

Given

- $\eta > 0$,
  - any countable collection $\mathcal{F} = \{f_q : \mathbb{R}^v \to \mathbb{R}\}$, each f_q has a nonvanishing derivative of some finite order on $[0, \eta]^v$.

Then there exists a set $E \subseteq [0, \eta]$ of full Minkowski dimension and Hausdorff dimension at least $1/(v - 1)$ such that $f_q(x_1, \cdots, x_q) \neq 0$ for each q and any v-tuple of distinct points $x_1, \cdots, x_v \in E$.

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Variants available when

- $\mathcal{F}$ consists of vector-valued functions,

- $\mathcal{F}$ possibly uncountable, containing functions with a common linearization.

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The notion of Fourier dimension

Definition

The Fourier dimension of a set $E \subseteq \mathbb{R}^n$ is defined as

$$\dim_{\mathbb{F}}(E) = \sup \left\{ \beta \in [0, n] \; \middle| \; \begin{array}{l} \exists \text{ a probability measure } \mu, \; \text{supp}(\mu) \subseteq E, \\ \sup_{\xi \in \mathbb{R}^n} \left| \widehat{\mu}(\xi) \right| \left(1 + |\xi| \right)^{\frac{\beta}{2}} < \infty \end{array} \right\}.$$

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- Sets for which equality holds are called *Salem sets*.

- Deterministic constructions for Salem sets exist, but are rare. They are however ubiquitous among random sets (Salem, Kahane).

Large Fourier dimension $\implies$ configurations? Take 1

Theorem (Shmerkin 2016)

For every $t \in (0, 1]$, there exists a Salem set of dimension t that avoids progressions.

Large Fourier dimension $\implies$ configurations? Take 2

Theorem (Łaba-P 2009)

Suppose $E \subseteq [0, 1]$ is a closed set which supports a probability measure μ with the following properties:

(a) (Ball condition)

$$\mu\big([x - \epsilon, x + \epsilon]\big) \leq C_1 \epsilon^\alpha \text{ for all } 0 < \epsilon \leq 1,$$

(b) (Fourier decay condition)

$$\big|\widehat{\mu}(\xi)\big| \leq C_2 |\xi|^{-\frac{\beta}{2}} \quad \text{for all } \xi \neq 0,$$

where $0 < \alpha < 1$ and $\frac{2}{3} < \beta \leq 1$. If $\alpha > 1 - \epsilon_0(C_1, C_2, \beta)$, then E contains a 3-term arithmetic progression.

- Many higher dimensional generalizations (Chan, Henriot, Łaba, P)

Where have all the points gone?

Metaprinciple: If a set is large, then the class of certain configurations lying in this space must be large as well, in terms of a natural measure defined on the class of such configurations.

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Falconer's conjecture

If $E \subseteq \mathbb{R}^n$, $\dim_{\mathbb{H}}(E) > \frac{n}{2}$, then the set of distances between pairs of points in E must have positive Lebesgue measure.

Large configuration spaces

For $E \subseteq \mathbb{R}^2$, let $T_2(E) = E^3/\sim$, where

$(a, b, c) \sim (a', b', c')$ if and only if $\triangle abc$ and $\triangle a'b'c'$ are congruent.

Large configuration spaces

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$(a, b, c) \sim (a', b', c')$ if and only if $\triangle abc$ and $\triangle a'b'c'$ are congruent.

Theorem (Greenleaf and Iosevich 2010)

Let $E \subseteq \mathbb{R}^2$ be a compact set with $\dim_{\mathbb{H}}(E) > \frac{7}{4}$. Then $T_2(E)$ has positive 3-dimensional measure.

Large configuration spaces

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$(a, b, c) \sim (a', b', c')$ if and only if $\triangle abc$ and $\triangle a'b'c'$ are congruent.

Theorem (Greenleaf and Iosevich 2010)

Let $E \subseteq \mathbb{R}^2$ be a compact set with $\dim_{\mathbb{H}}(E) > \frac{7}{4}$. Then $T_2(E)$ has positive 3-dimensional measure.

- Result says that triangles are abundant in sets of large Hausdorff dimension. However it does not (and in light of Maga's theorem cannot) guarantee the existence of (a similar copy of) a specific triangle.

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- Higher dimensional extensions: Erdogan, Hart, Grafakos, Greenleaf, Iosevich, Taylor, Liu, Mourgoglu, Palsson, P, ...

Plan

- Setup

- History

- Avoidance

- Existence

- **Abundance**

A different direction: Euclidean Ramsey theory

Suppose that

- $E \subseteq \mathbb{R}^n$ has positive upper density (with respect to Lebesgue measure), i.e.,

$$\limsup_{R \to \infty} \sup_{x \in \mathbb{R}^n} \frac{|E \cap B(x; R)|}{R^n} > 0.$$

- F is a non-degenerate $(k-1)$-dimensional simplex (i.e., a set of k points in general position).

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Theorem (Bourgain 1986)

If $k \leq n$, then E contains a translated and rotated copy of λF for all λ sufficiently large.

All sufficiently large copies of configurations (ctd)

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- (Graham 1994) The result is false for any **non-spherical** set in $\mathbb{R}^n$!

Norms and configurations?

Cook, Magyar, P 2016

- Fix any $1 < p < \infty$, $p \neq 2$.

- Then $\exists n_p < \infty$ such that $\forall n \geq n_p$,

- any set $A \subseteq \mathbb{R}^n$ of positive upper density contains a 3-term AP of every sufficiently large gap length, measured in ℓ^p norm.

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In other words, $\exists \lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $\exists x, x + y, x + 2y \in A$ such that $||y||_p = (\sum_{i=1}^{d} |y_i|^p)^{1/p} = \lambda$.

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Proving existence and/or abundance: Methodology

- Configuration counting functions, such as

$$\Lambda(f) = \iint f(x)f(x+y)f(x+2y) \, dy \, dx.$$

If $f = 1_A$, then $\Lambda(f)$ measures the number of 3 AP-s in A. If A is a sparse set supporting a measure, finer definitions of Λ are necessary.

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- Estimation of Λ:

  - Fourier-analytic methods, oscillatory integral estimates,

  - tools from additive combinatorics, such as Gowers norms,

  - tools from time-frequency analysis, e.g. bounds for multilinear singular integral operators such as the bilinear Hilbert transform.

Questions

- Same result for corners by Durcik, Kovač and Rimanić, ... other examples and/or counterexamples?

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- To what extent do configuration results rely on the underlying topology?

Thank you!