Differential Subordination under change of law

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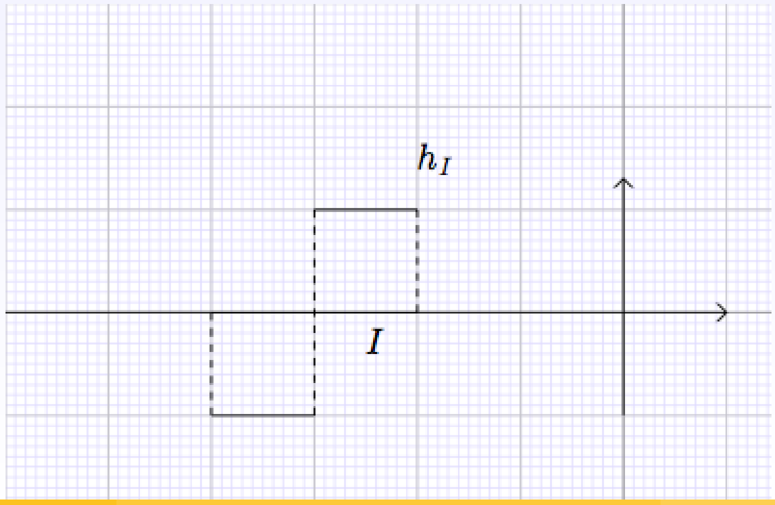
What this lecture is about

One may say that modern weighted theory started with the paper by NTV on the two weight problem for the 'dyadic predictable martingale multiplier'.

On $[0,1]$ endowed with Lebesgue measure and h_I the Haar system consider the L^2 isometry

$$T_\sigma : h_I \mapsto \sigma_I h_I, \sigma_I = \pm 1$$



What this lecture is about

From here on out, through the efforts of many, a rich weighted theory, discussing CZOs and beyond, drove and renewed our understanding of objects in harmonic analysis seen through this probabilistic framework.

This talk has a historic component, covering the early days of modern weighted theory and then comes back to present new results in probability theory, using and extending techniques developed between 1997 and as recent as 2015.

Nazarov-Treil-Volberg and Wittwer's theorem

Recall

$$T_\sigma : h_I \mapsto \sigma_I h_I.$$

NTV characterised boundedness in the two-weight setting in their 1997 paper, developing the Bellman technique. Applied to the one-weight case, a theorem by Wittwer states that uniformly in $|\sigma|_\infty \leq 1$,

$$\|T_\sigma\|_{L^2(w)\to L^2(w)} \lesssim Q_2(w)$$

where dyadic A_2 stands:

$$Q_2(w) = \sup_I \langle w\rangle_I \langle w^{-1}\rangle_I$$

where the supremum runs over all dyadic intervals.

Differential subordination 1

For each discrete time $0 \leq n$ consider the classical dyadic covering of size 2^{-n}, together with its generated sigma algebra.
This becomes a filtered probability space $([0,1], \mathcal{F}, dx)$

$$X_n = \mathbb{E}(f \mid \mathcal{F}_n) \text{ and } Y_n = \mathbb{E}(T_\sigma f \mid \mathcal{F}_n)$$

are a pair of martingales that are differentially subordinate.

$$|\Delta_n(T_\sigma f)| \leq |\sigma_{n-1}||\Delta_n(f)|$$

Here, $dX_n = \Delta_n(f) = \mathbb{E}(f \mid \mathcal{F}_n) - \mathbb{E}(f \mid \mathcal{F}_{n-1})$
$dY_n = \Delta_n(T_\sigma f) = \sigma_{n-1}\Delta_n(f)$

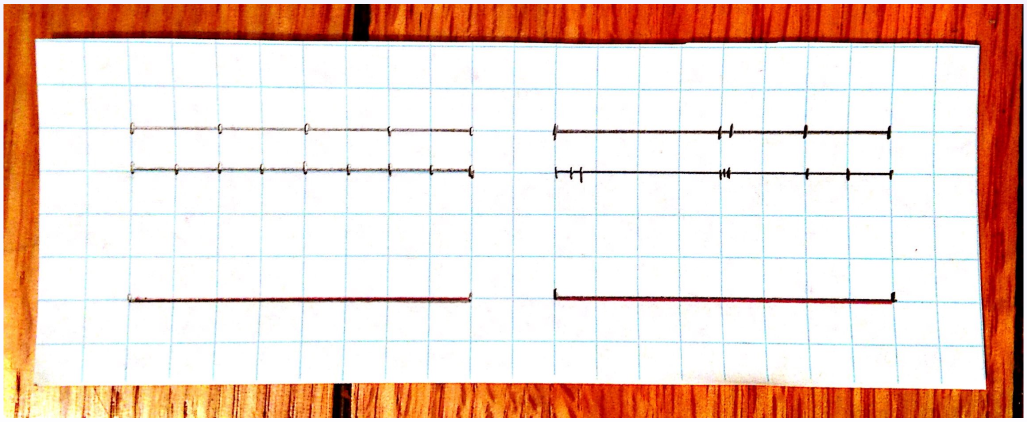
Note σ_{n-1} is measurable in $\mathcal{F}_{n-1}$ but perhaps not in $\mathcal{F}_n$. Such multipliers are called predictable.

A non-homogenous result

Predictable multipliers in general probability spaces with discrete in time filtrations have sharp weighted estimates using the martingale A_2 characteristic.

Thiele-Treil-Volberg 2015
Lacey 2015

independently



The Square bracket 1

We will change our notion of differential subordination according to Burkholder, away from predictable multipliers.

Square bracket, discrete filtration:

$$[X, X]_n = \sum_{k=1}^{n} (dX_k)^2$$

for example discrete random walk B: $[B, B]_n = \sum_{k=1}^{n} 1 = n$

Modern probability theory is concerned with filtered probability spaces with continuous time: (Ω, F, μ), for example the Brownian filtration.

The Square bracket 2

Square bracket and products (almost surely):

$$(XX)_n - (XX)_{n-1} = 2X_{n-1}(X_n - X_{n-1}) + (X_n - X_{n-1})^2 = 2X_{n-1}dX_n + (dX_n)^2$$

This can be generalised to continuous in time filtrations for the square bracket of the product of martingales. One obtains the bracket process:

$$[X, X] = X^2 - 2\int X_- dX$$

$[X, X]$ is also a limit of discrete sums $X_0^2 + \sum_i (X^{T_{i+1}^n} - X^{T_i^n})^2$ with T^n sequence of increasing stopping times. One defines $[X, Y]$ by polarisation. It is also the non-predictable compensator of X^2.

Differential Subordination 3

Y differentially subordinate to X if

$$[X, X]_t - [Y, Y]_t$$

is a non-negative and non-decreasing function of $t \geq 0$.

If the martingale has discontinuous paths (jumps) then this bracket differs from $\langle \cdot, \cdot \rangle$ in that subordination requires precise information at the instances of jumps.

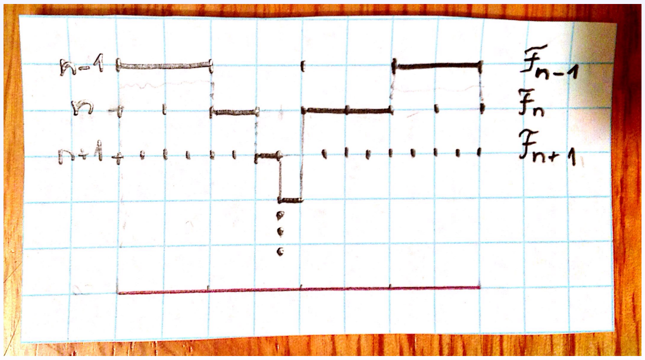
The difference of these two brackets is the reason behind the difficulty in giving precise estimates for discrete Calderon-Zygmund operators.

Weight characteristic

The martingale characteristic of the weight:

$$Q_2(w) = \sup_\tau \|\mathbb{E}(w_\tau w^{-1} \mid \mathcal{F}_\tau)\|_\infty = \sup_\tau \|w_\tau (w^{-1})_\tau\|_\infty$$

The supremum is over all adapted stopping times: random variables τ so that $\{\tau \geq t\} \in \mathcal{F}_t$. Here $\mathcal{F}_\tau$ is the stopped σ algebra i.e. the smallest σ algebra containing all càdlàg processes sampled at time τ.



Theorem1

Theorem (Domelevo, P.)

Y differentially subordinate to X (with values in Hilbert space) then sharp weighted estimates hold using the martingale characteristic of the weight:

$$\|Y\|_{L^2(w)} \lesssim Q_2(w)\|X\|_{L^2(w)}$$

Proof: delicate Bellman-only, Stochastic integrals, Ito integral formula

extends with correct bounds to L^p but requires a new maximal inequality (below)

Theorem2

Theorem (Domelevo, P.)

Y differentially subordinate to X (with values in Hilbert space) then maximal sharp weighted estimates hold using the martingale characteristic of the weight:

$$\|Y^*\|_{L^2(w)} \lesssim Q_2(w)\|X\|_{L^2(w)}$$

Proof: 'sparse' with continuous stopping time

For Y submartingale $Y^* = \sup_t |Y_t|$ for all events.

extends with correct bounds to L^p.

Theorem2'

Theorem (Domelevo, P.)

Setting $Y = X$ in the last theorem gives

$$\|X^*\|_{L^2(w)} \lesssim Q_2(w)\|X\|_{L^2(w)}$$

Proof: very easy change of measure calculation, using Doob's inequality.

To get the correct bound in L^p the sparse proof is not working for all exponents, this is why another argument makes sense here.

Even boundedness with any norm control was open (Lepingle, Bonami, PA Meyer). Classical proofs taken from analysis use the openness of the A_p conditions, which fails for stochastic processes in this generality (example by PA Meyer).

Wittwer's estimate: weak form

With $T_\sigma f = \sum_{I\in\mathcal{D}} \sigma_I \Delta_I f(x)$, Wittwer's theorem asserts that

$$\sup_\sigma \|T_\sigma\|_{w\to w} \leqslant CQ_2^1(w).$$

In its weak form, this becomes

$$\sup_\sigma |(T_\sigma f, g)| \leqslant CQ_2^1(w)\|f\|_w\|g\|_{w^{-1}}$$

or by choosing the worst σ with $J = [0,1]$:

$$\sup_\sigma |\sum_{I\in\mathcal{D}} \sigma_I (f,h_I)(g,h_I)| = \sum_{I\in\mathcal{D}} |(f,h_I)(g,h_I)| \leqslant Q_2(w)^1\|f\|_w\|g\|_{w^{-1}}$$

The classical Four sums proof

$h_I = \alpha_I^w h_I^w + \beta_I^w \chi_I^w$ and same for w^{-1} and get $2 * 2 = 4$ sums.

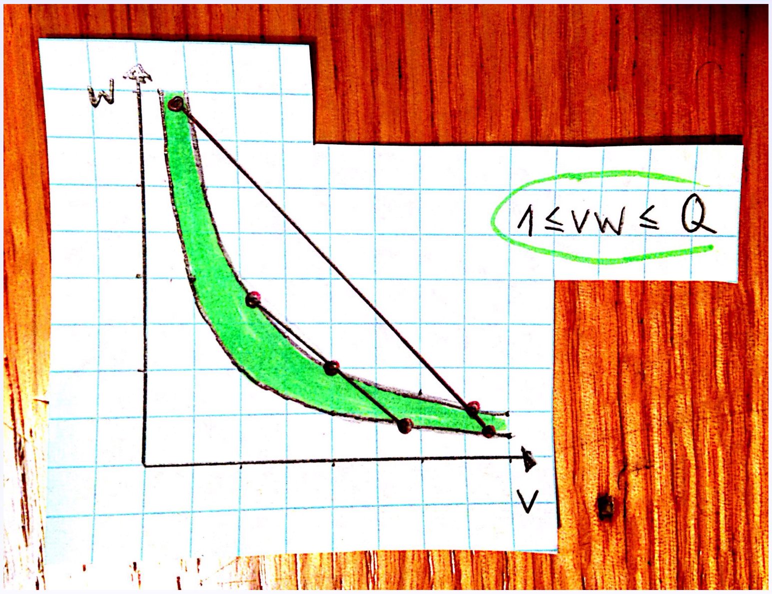
1) two cancellative: easy, just CS, use ONB property in weighted L^2
2) and 3) one cancellative one non-cancellative: medium, Carleson Lemma and Bellman
4) two non-cancellative: hard, bilinear Carleson Lemma and Bellman.

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} |(w, h_I)(w^{-1}, h_I)| \lesssim Q$$

$$B(v, w) = Q^{1/2}\sqrt{vw} - vw$$

when $1 \leq vw \leq Q$
Convexity plays a big role.

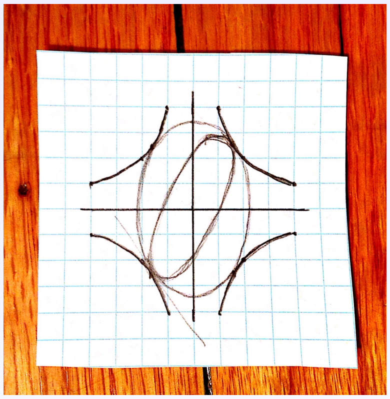
The classical Four sums proof, non-homogenous, discrete time



Taylor formula of the second order for the endpoints
Martingale property makes first order derivatives disappear
Hessian estimate only in the green region.

The classical Four sums proof, most general case

...turns into a 'one' sum (or integral) proof due to the difficulty related with the process $[Y, Z]$ vs $[X, Z]$, where one wishes to estimate Y by X through testing against Z.



Use of the very clever 'Ellipse Lemma' self improving estimates such as

$$(Av, v) \geq 2|v_1||v_2| \text{ to } (Av, v) \geq \tau|v_1|^2 + \tau^{-1}|v_2|^2$$

so as to make differentials of $[Y, Y]$, $[X, X]$ and $[Z, Z]$ appear.

The classical Four sums proof, most general case

The solution requires an **explicit** Bellman function of four variables for the entire problem.

The delicacy of the continuous time and the jumps cannot be separated in this problem. Many technical difficulties.

Closes a loop in the Bellman vs Burkholder theme!

Burkholder functions are those that give the norm estimates directly, built for the differential subordination condition. Bellman functions are those that give the norm estimate by duality and testing, requires Ellipse lemma.

The Burkholder function for the weighted problem is unknown - this is why this question remained open.

The Bellman function

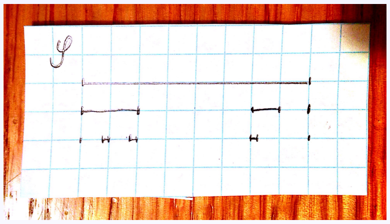
is very long...

but has other applications, see **Kamilia Dahmani**'s poster. Hers is a Riesz vector estimate in the Riemannian setting, including an underlying, possibly non-homogenous exponential measure (Gauss space).

Lacey's sparse proof

is very short...

1) Dominate $T_\sigma f(x) \lesssim \sum_{I \in \mathcal{S}} \langle |f| \rangle_I \chi_I$ through a stopping procedure.
Unweighted weak type bounds for various maximal operators make sure one does not stop very often.
Here $\mathcal{S}$ is a sparse collection that are the rare stopping intervals.



2) Use a well known argument via unweighted maximal functions to get the estimate.

Sparse with continuous stopping time

For those who know sparse proofs, here are the interesting changes:
1) stopping condition is not a number but a random variable
2) continuous stopping times give rise to new filtrations on the whole space $\mathcal{F}_t^n$
3) sparse condition is not as straightforward, measurable subsets of iterate $\mathcal{F}_0^n$'s play a role.

does not need predictable multipliers.

Easiest stochastic integral for Hilbert transform

Let $f(x)$ a smooth function defined on the unit circle $\mathbb{T}$. Let $\tilde{f}(x,y)$ its harmonic extension in the unit disc. Let $B_t = (x_t, y_t)$ the 2–dimensional Brownian motion started at $B_0 = (0,0)$, and stopped at time τ_1 when the random walk hits the boundary $\tau_1 = \inf(t; (x_t, y_t) \in \mathbb{T})$. Then $f(B_{t\wedge\tau_1})$ is a martingale and Itô's formula states

$$\tilde{f}(B_{t\wedge\tau_1}) \quad = \quad \tilde{f}(B_0) + \int_0^{t\wedge\tau_1} \nabla\tilde{f}(B_{s-}) \cdot \mathrm{d}B_s$$

Due to harmonicity Ito's formula appears short.

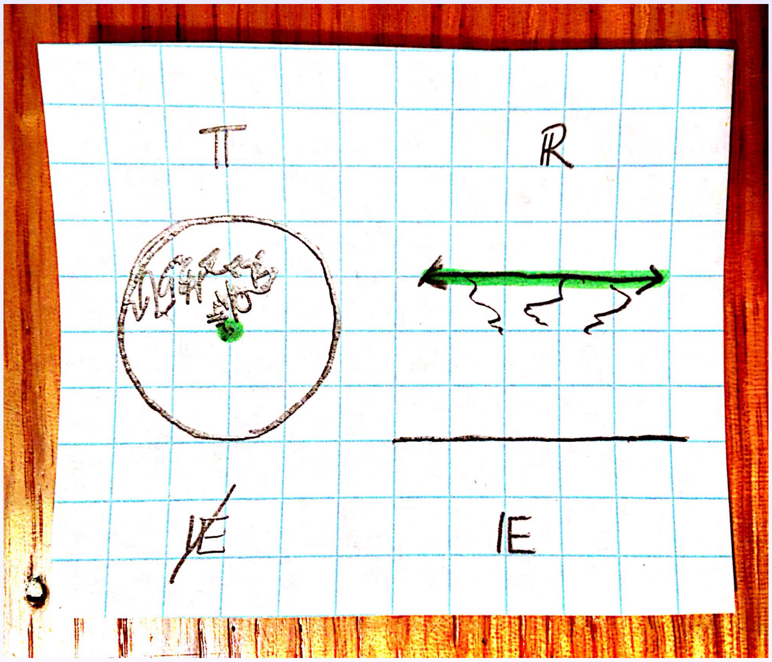
Easiest stochastic integral for Hilbert transform

We have similarly, setting $g = Hf$

$$\begin{aligned} \tilde{g}(B_{t\wedge\tau_1}) &= \tilde{g}(B_0) + \int_0^{t\wedge\tau_1} \nabla\tilde{g}(B_{s-})\cdot \mathrm{d}B_s \\ &= \int_0^{t\wedge\tau_1} \nabla^{\perp}\tilde{f}(B_{s-})\cdot \mathrm{d}B_s \end{aligned}$$

where we observed that $\tilde{g}(B_0) = 0$, and used analyticity of $\tilde{f} + i\tilde{g}$ which implies through Cauchy–Riemann relations that $\nabla\tilde{g} = \nabla^{\perp}\tilde{f}$, where $\nabla^{\perp} = (-\partial_y, \partial_x)$. Clearly $\tilde{g}(B_{t\wedge\tau_1})$ is a martingale differentially subordinate to $\tilde{f}(B_{t\wedge\tau_1})$.

Other cases



Gundy-Varopoulos H and R in $\mathbb{R}^n$
Arcozzi-Domelevo-P **discrete** Hilbert transform and Riesz transforms of the second order, jump processes and Brownian Motion.

Beurling-Ahlfors

Here is a motivation, the solution to a long standing regularity problem in PDE, solved through the optimal weighted norm estimate of a classical singular integral operator, that itself relied on an optimal martingale estimate under a change of law.

$$Tf(z) = p.v. \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dA(w)$$

T exchanges ∂ and $\bar{\partial}$

Beltrami equation

$$\partial f - \mu \bar{\partial} f = 0$$

with $\|\mu\|_\infty = k < 1$ relates to the operator $I - \mu T$ applied to ∂f.

$K = (k+1)/(k-1)$ ratio of axes of infinitesimal ellipses, images of disks under f, the homeomorphic solution.

Beurling-Ahlfors

What is the minimal requirement of the type $f \in W^{1,?}$ which guarantees any solution for any μ with $\|\mu\|_\infty = k < 1$ self improves to $W^{1,2}$ (hence is continuous)?

Answer: $p \geq 1 + k$

Astala - Iwaniec - Saksman (strict ineq)
P. - Volberg (borderline case)

Beurling-Ahlfors

this happens if $I - \mu T$ injective in L^p.

invertability in L^p of $I - \mu T$ when $p \in (k+1, 1+1/k)$
injectivity when $p = k+1$
dense range when $p = 1+1/k$.

Beurling-Ahlfors

Using a special weight w related to the equation, then 'any' bound for

$$T : L^p(w) \to L^p(w)$$

gives the desired result in the open interval of p.

The solution of the borderline case $p = 1 + 1/k$ required an estimate 'linear' in $Q_p(w)$ such as

$$\|T\|_{L^p(w)\to L^p(w)} \leq C_p Q_p(w)^1,$$

because at the borderline, the special weight fails to belong to A_p.