Two-Weight Inequalities for Commutators with Calderón-Zygmund Operators

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Joint work with Brett D. Wick and Michael Lacey

Washington University in St. Louis

MSRI Workshop
Connections for Women - Harmonic Analysis

Outline

Introduction

Starting point: Coifman, Rochberg and Weiss, *Factorization theorems for Hardy spaces in several variables*, 1976

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- Riesz transforms
- Calderón-Zygmund Operators

$$Tf(x) := \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$$

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$$[b, T]f := b(Tf) - T(bf)$$

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- $H^1(\mathbb{R}^n)$ – $BMO(\mathbb{R}^n)$ Duality (Fefferman, 1971)

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Upper Bound:

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Lower Bound:

$$\|b\|_{BMO} \lesssim \sum_{j=1}^{n} \|[b, R_j] : L^p \to L^p\|.$$

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- Two-weight Inequalities: $T : L^p(\mu) \to L^p(\lambda)$

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- $H : L^p(w) \to L^p(w) \Leftrightarrow w \in A_p$
- A_2 weights:

$$[w]_{A_2} := \sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q$$

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- What if $\mu \neq \lambda$?

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Bloom (1985)

$[b, H]\colon L^p \to L^p$

bounded

$\Longleftrightarrow$

$b \in BMO$

$$\|b\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q \left| b(x) - \langle b \rangle_Q \right| dx$$

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$\Longleftrightarrow$

$b \in BMO(\nu)$

$$\|b\|_{BMO} \coloneqq \sup_Q \frac{1}{|Q|} \int_Q \left| b(x) - \langle b \rangle_Q \right| dx$$

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$[b, H] \colon L^p(\mu) \to L^p(\lambda)$

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$$\nu := \mu^{1/p} \lambda^{-1/p}$$

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$[b, H]: L^p(\mu) \to L^p(\lambda)$ bounded $\iff$ $b \in BMO(\nu)$

- Extend to all CZO's T on $\mathbb{R}^n$

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$[b, H]: L^p(\mu) \rightarrow L^p(\lambda)$

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⟺

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- Extend to all CZO's T on $\mathbb{R}^n$
- Long-term: Extend to *multiparameter setting*

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- Dyadic approach

Outline

CRW:

Upper Bound:

$$\|[b, T] : L^p \to L^p\| \lesssim \|b\|_{BMO}$$

Lower Bound:

$$\|b\|_{BMO} \lesssim \sum_{j=1}^{n} \|[b, R_j] : L^p \to L^p\|.$$

Main Results (H., Lacey, Wick):

Upper Bound:

$$\|[b, T] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}$$

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Upper Bound: Strategy

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I. Use a Representation Theorem to reduce the problem to bounding

$$[b, \text{Dyadic Shift}]$$

Upper Bound: Strategy

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II. Bound:

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Dyadic Grids:

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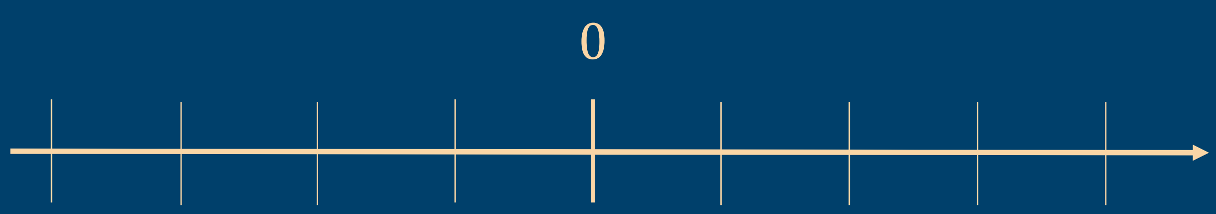
Dyadic Grids: $\mathcal{D}_0$



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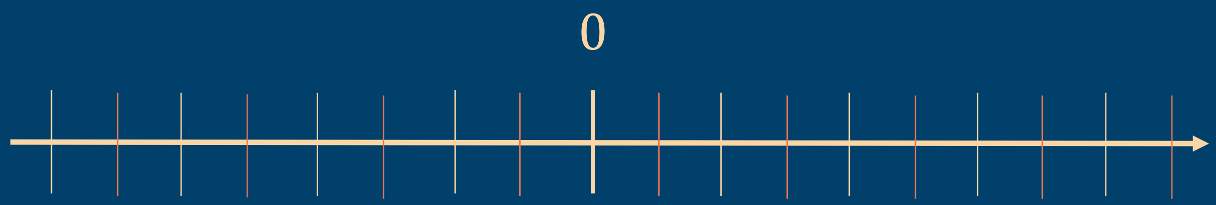

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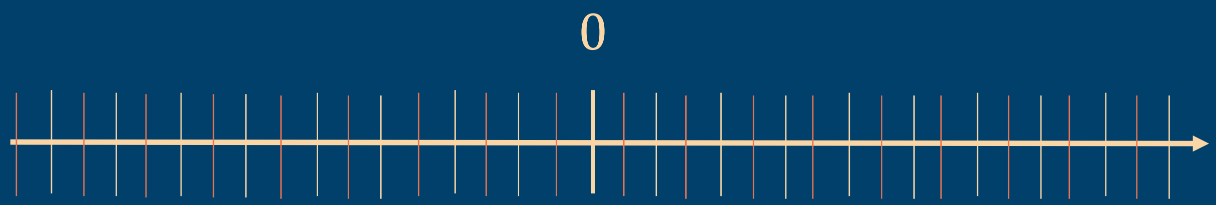

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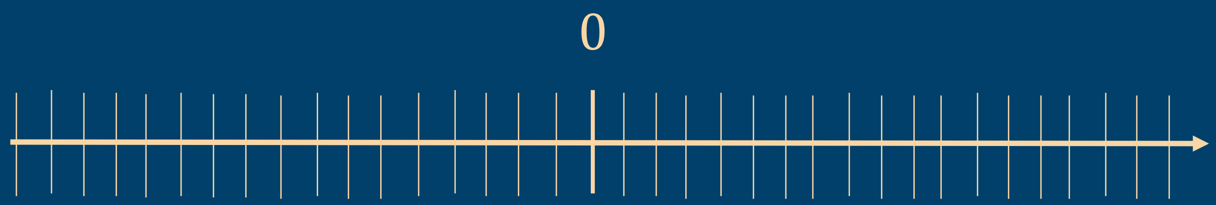

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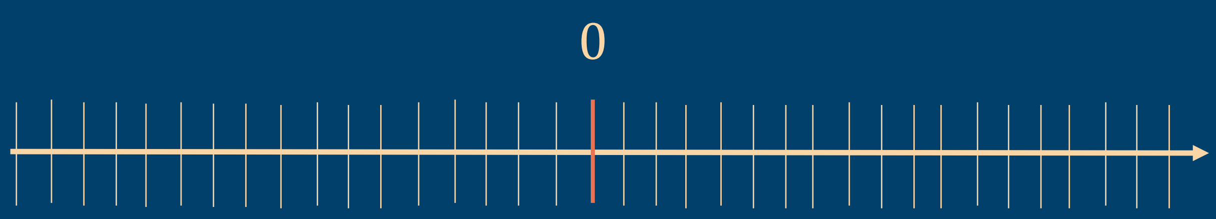

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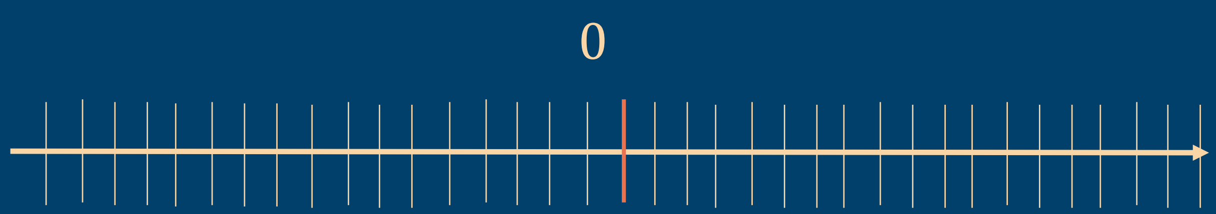


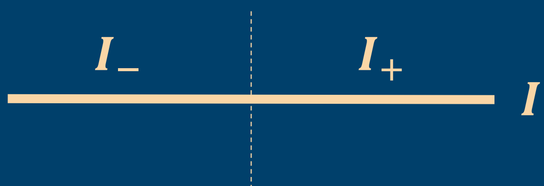
Upper Bound: Strategy

I. Use a Representation Theorem to reduce the problem to bounding

$$[b, \text{Dyadic Shift}]$$

Haar Functions: $I \in \mathcal{D}$

$$h_I := \frac{1}{\sqrt{|I|}} \left(\mathbb{1}_{I_-} - \mathbb{1}_{I_+} \right)$$



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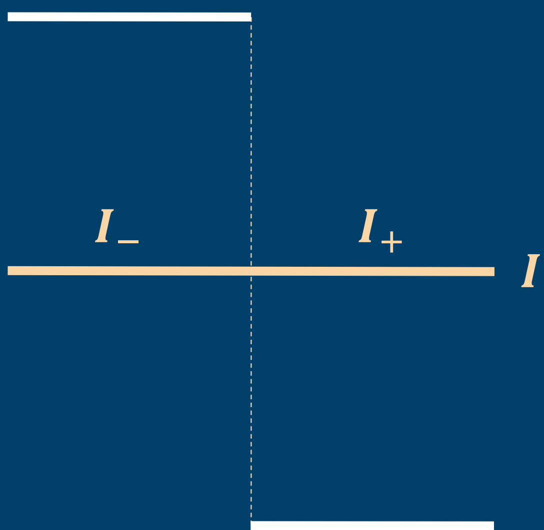

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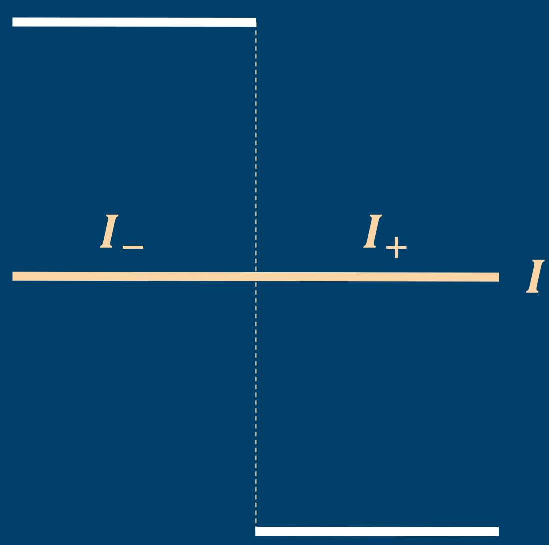

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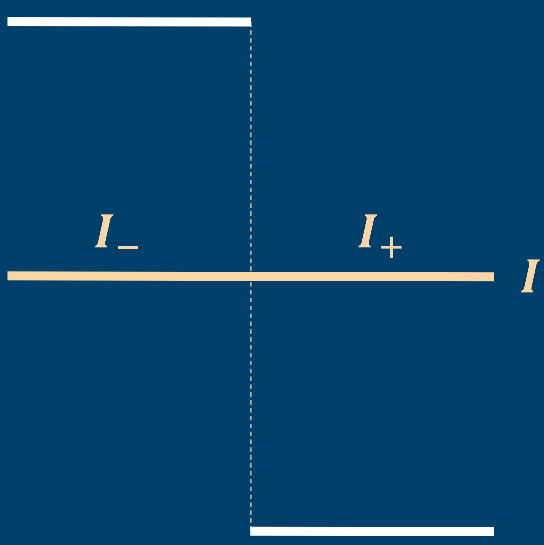
$$f = \sum_{I \in \mathcal{D}} \widehat{f}(I) h_I$$



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Petermichl's Dyadic Shift:

$$\text{Ш}_{\omega} f := \frac{1}{\sqrt{2}} \sum_{I \in \mathcal{D}_{\omega}} \widehat{f}(I) \left(h_{I_-} - h_{I_+} \right).$$

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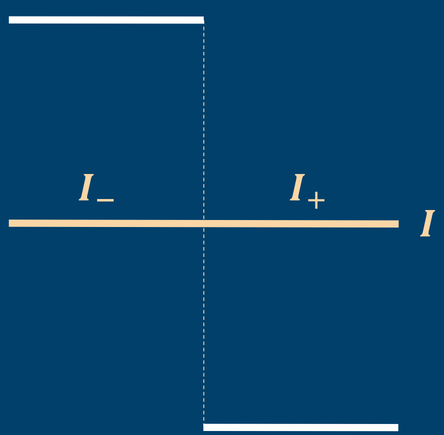

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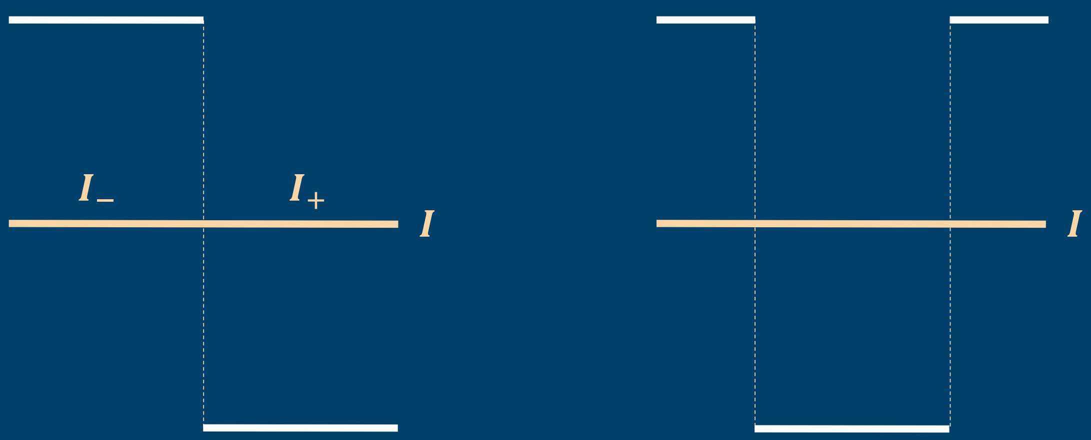

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For general CZOs on $\mathbb{R}^n$:

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For general CZOs on $\mathbb{R}^n$: Hytönen Representation Theorem (2011).

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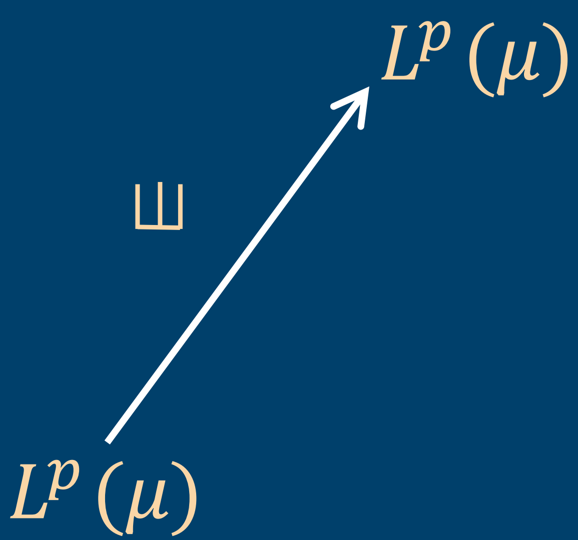

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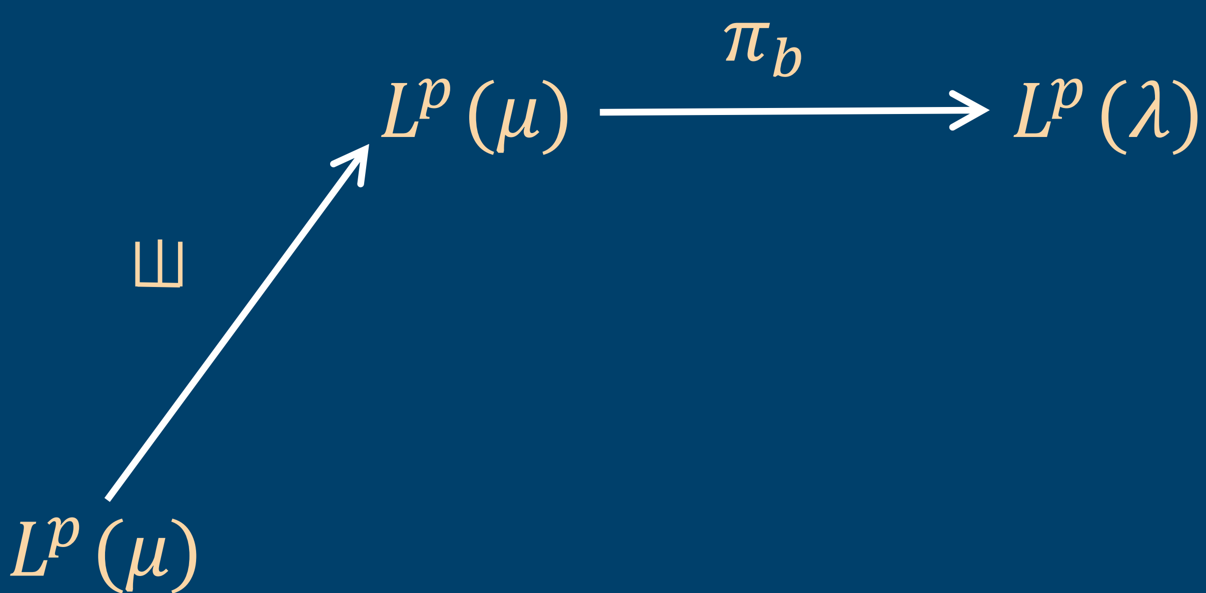

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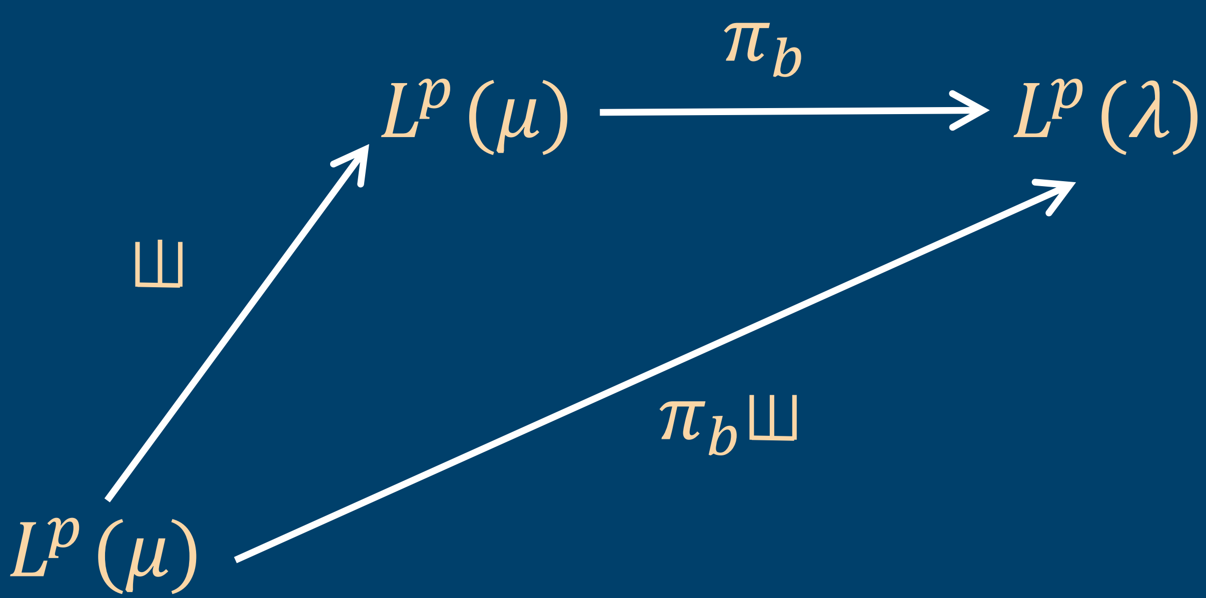

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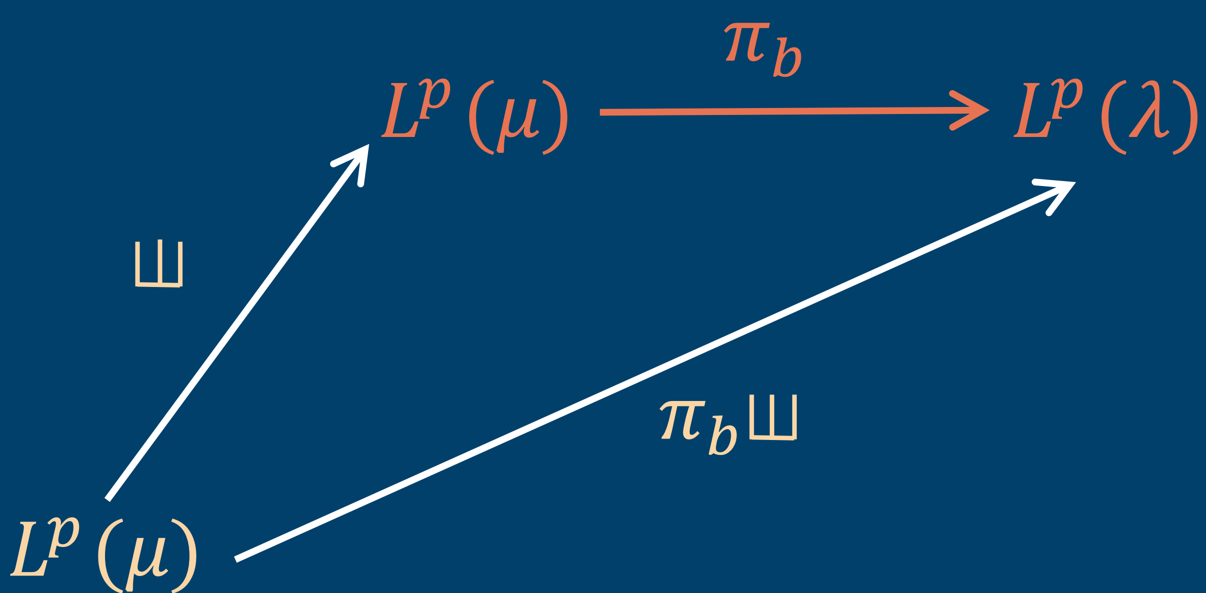

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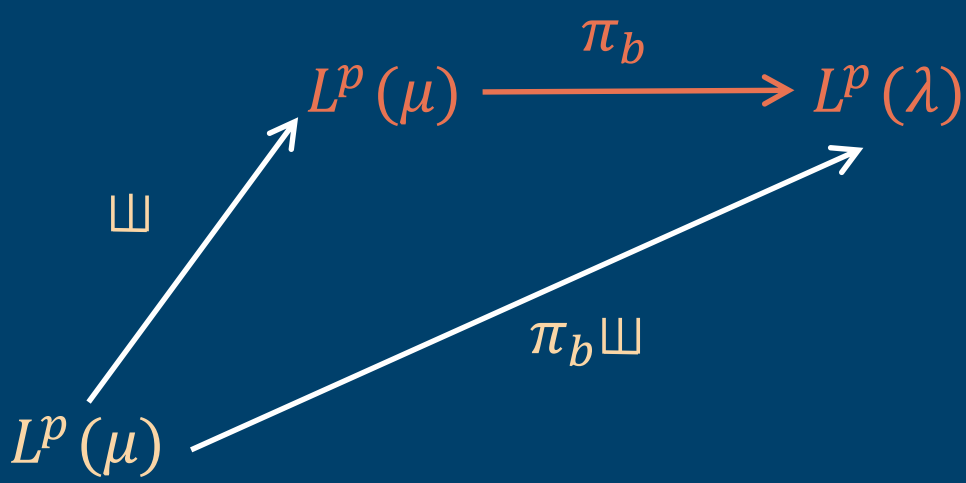

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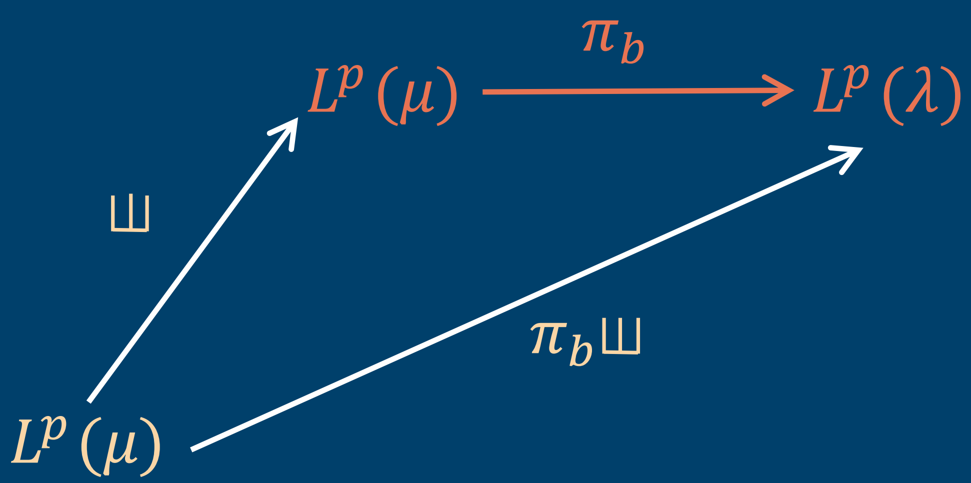

- Reduce to one-weight maximal and square function estimates!

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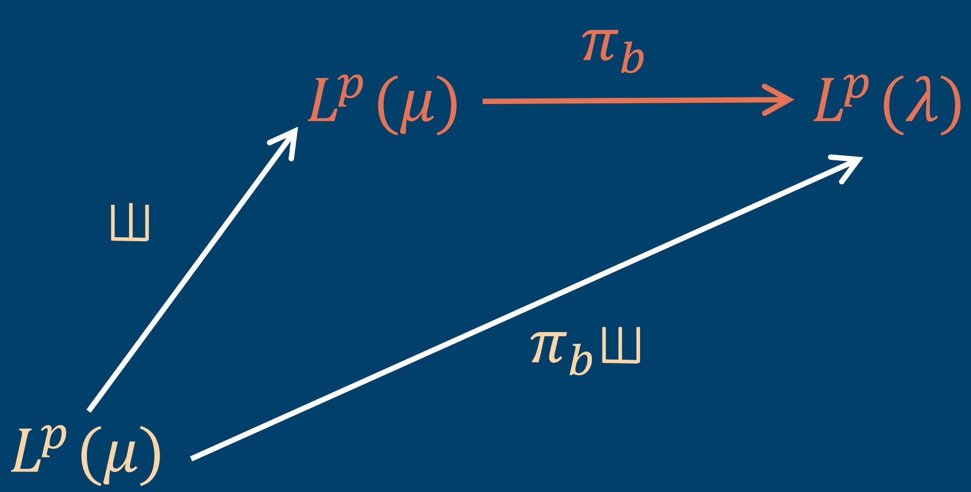

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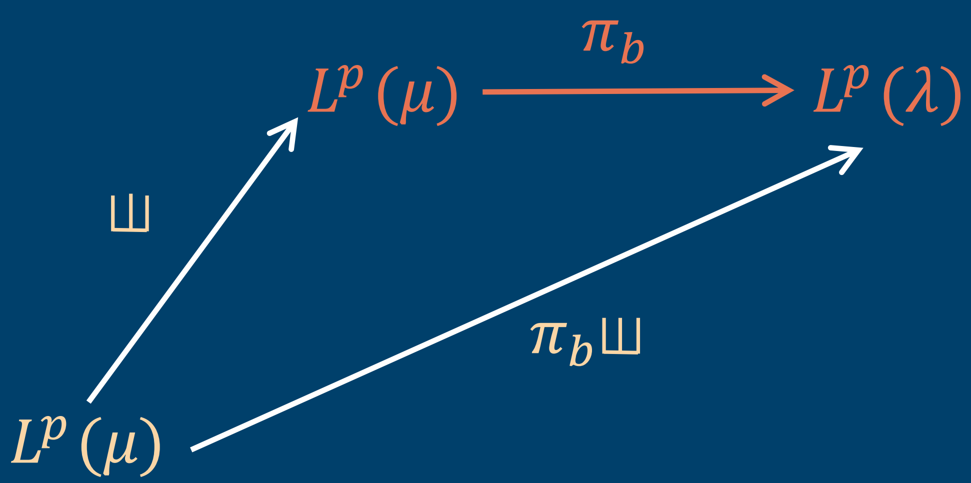

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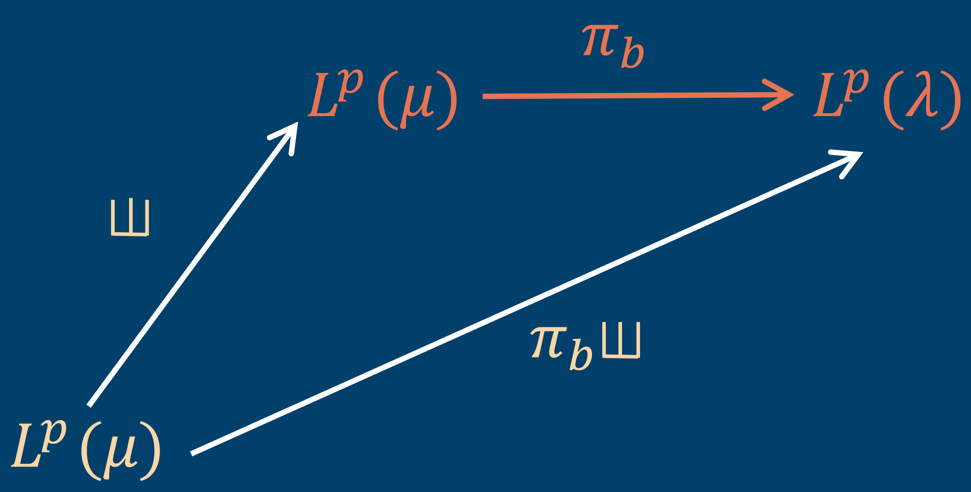

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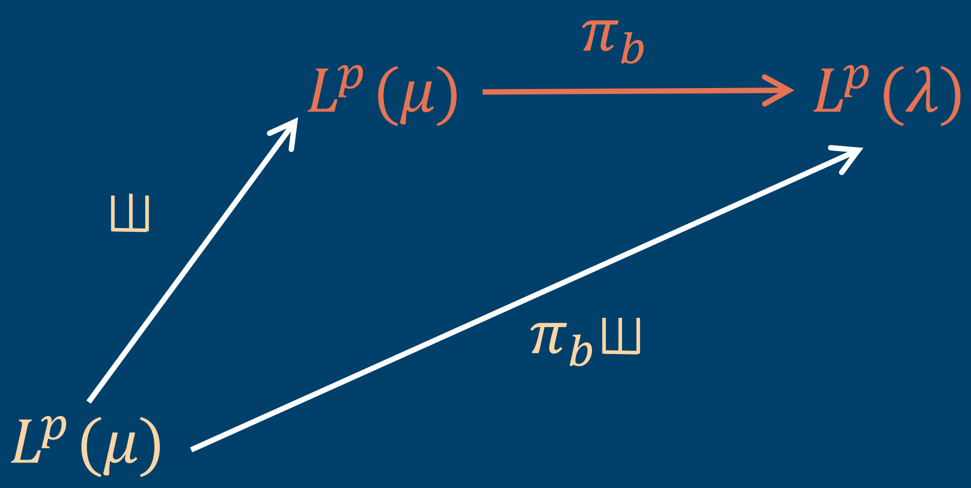

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- $\nu = \mu^{1/p}\lambda^{1/p} \in A_2$!!!

Outline

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$$\cong$$

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$\cong$ Muckenhoupt & Wheeden ('75)

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