Two-Weight Inequalities for Commutators with Calderón-Zygmund Operators

Irina Holmes Joint work with Brett D. Wick and Michael Lacey

Washington University in St. Louis

MSRI Workshop Connections for Women - Harmonic Analysis

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Outline

Introduction

Bloom's Result

Main Results

Upper Bound

Lower Bound: Key Idea

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 \triangleright Characterize the norm of the commutator $[b, T]$, where T is a CZO, acting $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, in terms of the BMO norm of *b*.

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Recall:

 \blacktriangleright Hilbert transform

$$
Hf(x) := p. \ \mathsf{v} \cdot \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy
$$

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 \blacktriangleright Riesz transforms

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$$

- $\overline{\triangleright}$ Riesz transforms
- \blacktriangleright Calderón-Zygmund Operators

$$
\mathcal{T}f(x) := \int_{\mathbb{R}^n} K(x,y)f(y) \, dy
$$

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 $[b, T]$ *f* := *b*(*Tf*) – *T*(*bf*)

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Bounded Mean Oscillation

$$
||b||_{BMO} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
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 \blacktriangleright *H*¹(\mathbb{R}^n) – *BMO*(\mathbb{R}^n) Duality (Fefferman, 1971)

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Upper Bound:

 $\|[b, T] : L^p \to L^p\| \lesssim \|b\|_{BMO}$

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Upper Bound:

$$
\|[b, T] : L^p \to L^p\| \lesssim \|b\|_{BMO}
$$

Lower Bound:

$$
||b||_{BMO}\lesssim \sum_{j=1}^n \|[b,R_j]:L^p\to L^p\|.
$$

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 \blacktriangleright Weight: non-negative, locally integrable function *w* on \mathbb{R}^n .

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 \blacktriangleright *L*^{*p*}(*w*): $\int |f(x)|^p w(x) dx$

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- \blacktriangleright *L*^{*p*}(*w*): $\int |f(x)|^p dw$
- \triangleright One-weight Inequalities: $T: L^p(w) \to L^p(w)$

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- ▶ One-weight Inequalities: $T: L^p(w) \to L^p(w)$ mostly \checkmark

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 \triangleright Two-weight Inequalities: $T: L^p(\mu) \to L^p(\lambda)$

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- \rightarrow One-weight Inequalities: $T: L^p(w) \rightarrow L^p(w)$ mostly \checkmark
- \triangleright Two-weight Inequalities: $T: L^p(\mu) \to L^p(\lambda)$ much harder!

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Recall:

 \blacktriangleright *A_p* weights:

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[w]_{A_p}:=\sup_Q \left\langle w\right\rangle_Q \left\langle w^{1-q}\right\rangle_Q^{p-1}
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Muckenhoupt, Hunt, Wheeden (1970's)

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- Muckenhoupt, Hunt, Wheeden (1970's)
- $M: L^p(w) \to L^p(w) \Leftrightarrow w \in A_p$

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- Muckenhoupt, Hunt, Wheeden (1970's)
- $H: L^p(w) \to L^p(w) \Leftrightarrow w \in A_p$
- \blacktriangleright *A*₂ weights:

$$
[w]_{A_2} := \sup_Q \langle w \rangle_Q \langle w^{-1} \rangle_Q
$$

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Recall:

 \triangleright OK in the one-weight case $\mu = \lambda$.

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Recall:

- \triangleright OK in the one-weight case $\mu = \lambda$.
- ► What if $\mu \neq \lambda$?

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Recall:

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- ► What if $\mu \neq \lambda$? Bloom!

Outline

Introduction

Bloom's Result

Main Results

Upper Bound

Lower Bound: Key Idea

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$$
[b, H]: L^p \to L^p
$$
 bounded
$$
b \in BMO
$$

$$
||b||_{BMO} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
$$

 $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}(\mathcal{A})$ **◆ロト → 伊ト → ミト → ミト** 重

$$
[b, H]: Lp(w) \rightarrow Lp(w)
$$
 bounded $b \in BMO$

$$
||b||_{BMO} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
$$

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$$
[b, H]: L^{p}(\mu) \to L^{p}(\lambda)
$$

bounded

$$
b \in BMO
$$

$$
||b||_{BMO} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
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$$
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 bounded

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||b||_{BMO} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
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[b, H]: L^{p}(\mu) \to L^{p}(\lambda)
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bounded

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4 ロ ▶ - 4 印 ▶ - 4 至 ▶ - 4 至 ▶ $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}(\mathcal{A})$ 目

$$
[b, H]: L^{p}(\mu) \to L^{p}(\lambda)
$$

bounded

$$
b \in BMO(\nu)
$$

$$
v := \mu^{1/p} \lambda^{-1/p}
$$

$$
||b||_{BMO} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
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[b, H]: Lp(\mu) \rightarrow Lp(\lambda)
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b \in BMO(\nu)
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v := \mu^{1/p} \lambda^{-1/p}
$$

$$
||b||_{BMO(v)} := \sup_{Q} \frac{1}{|Q|} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
$$

K ロ ▶ | K ⑦ ▶ | K 夏 ▶ | K 夏 ▶ $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}(\mathcal{A})$ 重

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[b, H]: L^{p}(\mu) \to L^{p}(\lambda)
$$

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$$
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$$
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||b||_{BMO(\nu)} := \sup_{Q} \frac{1}{\nu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
$$

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$$
[b, H]: L^{p}(\mu) \to L^{p}(\lambda)
$$

bounded

$$
b \in BMO(\nu)
$$

$$
\nu := \mu^{1/p} \lambda^{-1/p}
$$

$$
||b||_{BMO(\nu)} := \sup_{Q} \frac{1}{\nu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q}| dx
$$

 $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}(\mathcal{A})$ 重

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 \triangleright Extend to all CZO's T on \mathbb{R}^n

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 \triangleright Extend to all CZO's T on \mathbb{R}^n > Long-term: Extend to multiparameter setting

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- \triangleright Extend to all CZO's T on \mathbb{R}^n
- ! Long2term:'Extend'to'*multiparameter setting*
- ▶ Dyadic approach

Outline

Introduction

Bloom's Result

Main Results

Upper Bound

Lower Bound: Key Idea

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CRW:

Upper Bound:

$$
\|[b, T] : L^p \to L^p\| \lesssim \|b\|_{BMO}
$$

Lower Bound:

$$
||b||_{BMO}\lesssim \sum_{j=1}^n \|[b,R_j]:L^p\to L^p\|.
$$

Main Results (H., Lacey, Wick):

Upper Bound:

$$
\|[b, T]: L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}
$$

Lower Bound:

$$
||b||_{BMO}\lesssim \sum_{j=1}^n ||[b,R_j] : L^p\to L^p||.
$$

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Main Results (H., Lacey, Wick):

Upper Bound:

$$
\|[b, T] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}
$$

Lower Bound:

$$
||b||_{BMO}\lesssim \sum_{j=1}^n ||[b,R_j] : L^p\to L^p||.
$$

$$
\nu := \mu^{\frac{1}{p}} \lambda^{-\frac{1}{p}}
$$

$$
\|b\|_{BMO(\nu)} := \sup_{Q} \frac{1}{\nu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q} | dx
$$

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Main Results (H., Lacey, Wick):

Upper Bound:

$$
\|[b, T] : L^p(\mu) \to L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}
$$

Lower Bound:

$$
||b||_{BMO(\nu)} \lesssim \sum_{j=1}^n ||[b, R_j] : L^p(\mu) \to L^p(\lambda)||.
$$

$$
\nu:=\mu^{\frac{1}{p}}\lambda^{-\frac{1}{p}}\\\|b\|_{BMO(\nu)}:=\sup_{Q}\frac{1}{\nu(Q)}\int_{Q}\left|b(x)-\left\langle b\right\rangle_{Q}\right|dx
$$

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Outline

Introduction

Bloom's Result

Main Results

Upper Bound

Lower Bound: Key Idea

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$\Vert [b, T] : L^p(\mu) \to L^p(\lambda) \Vert \lesssim \Vert b \Vert_{BMO(\nu)}$

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$\Vert [b, T] : L^p(\mu) \to L^p(\lambda) \Vert \lesssim \Vert b \Vert_{BMO(\nu)}$

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

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$\|[\overline{b},\overline{T}]: L^p(\mu) \to L^p(\overline{\lambda})\| \lesssim \|b\|_{BMO(\nu)}$

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

II. Bound:

 $k[\![b,\overline{D}]\]$ yadic Shift $]: L^p(\mu) \rightarrow \overline{L^p(\lambda)}\] \lesssim \|b\|_{BMO(\nu)}$

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

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Dyadic Grids:

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Dyadic Grids: \mathcal{D}_0

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Dyadic Grids: \mathcal{D}_0

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Dyadic Grids: \mathcal{D}_0

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Dyadic Grids: \mathcal{D}_0

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Dyadic Grids: \mathcal{D}_0

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Dyadic Grids: \mathcal{D}_{ω}

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I. Use a Representation Theorem to reduce the problem to bounding

[b, Dyadic Shift]

Haar Functions: $I \in \mathcal{D}$

$$
h_I:=\frac{1}{\sqrt{|I|}}\left(\mathbb{1}_{I_-}-\mathbb{1}_{I_+}\right)
$$

I. Use a Representation Theorem to reduce the problem to bounding

[b, Dyadic Shift]

Haar Functions: $I \in \mathcal{D}$

$$
\mathbf{\mathsf{h}}_{\boldsymbol{\mathsf{I}}}:=\frac{1}{\sqrt{|\boldsymbol{\mathsf{I}}|}}\left({\mathbbm{1}}_{\boldsymbol{\mathsf{I}}_-}-{\mathbbm{1}}_{\boldsymbol{\mathsf{I}}_+}\right)
$$

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Haar Functions:

 ${h_1 : I \in \mathcal{D}}$ = onb for L^2 *.*

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Haar Functions:

 $f = \sum$ *I*2*D f* b(*I*)*h^I*

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Petermichl's Dyadic Shift:

$$
\mathrm{III}_{\omega}f:=\frac{1}{\sqrt{2}}\sum_{I\in\mathcal{D}_{\omega}}\widehat{f}(I)\left(h_{I_-}-h_{I_+}\right).
$$

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Petermichl's Dyadic Shift:

$$
\mathrm{III}_{\omega}f:=\frac{1}{\sqrt{2}}\sum_{I\in\mathcal{D}_{\omega}}\widehat{f}(I)\left(h_{I_-}-h_{I_+}\right).
$$

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Petermichl's Dyadic Shift:

$$
\mathrm{III}_{\omega} f := \frac{1}{\sqrt{2}} \sum_{l \in \mathcal{D}_{\omega}} \widehat{f}(l) \left(h_{l-} - h_{l_+} \right).
$$

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Petermichl's Dyadic Shift:

$$
\mathrm{III}_{\omega} f := \frac{1}{\sqrt{2}} \sum_{I \in \mathcal{D}_{\omega}} \widehat{f}(I) \left(h_{I_-} - h_{I_+} \right).
$$

Petermichl (2000): $\boxed{Hf = c\mathbb{E}_{\omega}\left(\mathrm{III}_{\omega}f\right)}$

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Petermichl's Dyadic Shift:

$$
\mathrm{III}_{\omega}f:=\frac{1}{\sqrt{2}}\sum_{I\in\mathcal{D}_{\omega}}\widehat{f}(I)\left(h_{I_-}-h_{I_+}\right).
$$

Petermichl (2000): $\boxed{Hf = c\mathbb{E}_{\omega}\left(\mathrm{III}_{\omega}f\right)}$

$$
\Rightarrow \left[\overline{[b,H]f=c\mathbb{E}_{\omega}\left([b,\mathrm{III}_{\omega}]f \right) }\right]
$$

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

Petermichl's Dyadic Shift:

$$
\amalg_{\omega} f := \frac{1}{\sqrt{2}} \sum_{I \in \mathcal{D}_{\omega}} \widehat{f}(I) \left(h_{I_{-}} - h_{I_{+}} \right).
$$

Petermichl (2000): $\left| Hf = c \mathbb{E}_{\omega} \left(\mathrm{III}_{\omega} f \right) \right|$

$$
\Rightarrow \left[\overline{[b,H]f} = c \mathbb{E}_{\omega} \left(\overline{[b,\text{III}_\omega]f} \right) \right]
$$

 $\Vert [b,\text{III}_{\omega}] : L^p(\mu) \to L^p(\lambda) \Vert \lesssim \Vert b \Vert_{BMO(\nu)}$

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I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

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For general CZOs on R*n*:

I. Use a Representation Theorem to reduce the problem to bounding

[*b,* Dyadic Shift]

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For general CZOs on Rⁿ: Hytönen Representation Theorem (2011).

 $I.$ Bound: $\| [b, Dy]$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \lesssim \|b\|_{BMO(\nu)}$

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 IIB Bound: $\|[b, b]$ Dyadic Shift] : $L^p(\mu) \to L^p(\lambda)$ $\|\leq \|b\|_{BMO(\nu)}$ Paraproducts:

$$
\pi_b f := \sum_l \widehat{b}(l) \langle f \rangle_l h_l \quad \pi_b^* f := \sum_l \widehat{b}(l) \widehat{f}(l) \frac{\mathbb{1}_l}{|l|}
$$

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 $I = E$ Bound: $\| [b, Dy$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \leq \| b \|_{BMO(\nu)}$ Paraproducts:

$$
\pi_b f := \sum_I \widehat{b}(I) \langle f \rangle_I h_I \qquad \pi_b^* f := \sum_I \widehat{b}(I) \widehat{f}(I) \frac{\mathbb{1}_I}{|I|}
$$

$$
bf = \pi_b f + \pi_b^* f + \pi_f b
$$

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 IIB Bound: $\|[b]$ Dyadic Shift] : $L^p(\mu) \to L^p(\lambda)$ $\|\leq \|b\|_{BMO(\nu)}$ Paraproducts:

$$
\pi_b f := \sum_I \widehat{b}(I) \langle f \rangle_I h_I \qquad \pi_b^* f := \sum_I \widehat{b}(I) \widehat{f}(I) \frac{\mathbb{1}_I}{|I|}
$$

$$
bf = \pi_b f + \pi_b^* f + \pi_f b
$$

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 $[b, \text{III}]$ *f* = $b(\text{III} f) - \text{III}(bf)$

 $I = E$ Bound: $\| [b, Dy$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \leq \| b \|_{BMO(\nu)}$ Paraproducts:

$$
\pi_b f := \sum_I \widehat{b}(I) \langle f \rangle_I h_I \qquad \pi_b^* f := \sum_I \widehat{b}(I) \widehat{f}(I) \frac{\mathbb{1}_I}{|I|}
$$

$$
bf = \pi_b f + \pi_b^* f + \pi_f b
$$

$$
[b, \text{III}]f = b(\text{III}f) - \text{III}(bf)
$$

=
$$
(\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f
$$

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 $\mathcal{P}(\mathcal{A}) \subset \mathcal{P}(\mathcal{A})$

 IIB Bound: $\|[b, b]$ Dyadic Shift] : $L^p(\mu) \to L^p(\lambda)$ $\|\leq \|b\|_{BMO(\nu)}$ Paraproducts:

$$
\pi_b f := \sum_I \widehat{b}(I) \langle f \rangle_I h_I \qquad \pi_b^* f := \sum_I \widehat{b}(I) \widehat{f}(I) \frac{\mathbb{1}_I}{|I|}
$$

$$
bf = \pi_b f + \pi_b^* f + \pi_f b
$$

$$
[b, \text{III}]f = b(\text{III}f) - \text{III}(bf)
$$

= $(\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f$
+ $(\pi_{\text{III}f}b - \text{III}\pi_f b)$

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II. Bound: $\|[b, Dyadic\]$ Shift] : $L^p(\mu) \rightarrow L^p(\lambda)\| \lesssim \|b\|_{BMO(\nu)}$

 $[b, \text{III}]f = (\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f + (\pi_{\text{III}f}b - \text{III}\pi_f b)$

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 $I.$ Bound: $\| [b, Dy]$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \lesssim \|b\|_{BMO(\nu)}$

$$
[b, \text{III}]f = \underbrace{(\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f}_{\checkmark} + (\pi_{\text{III}f}b - \text{III}\pi_f b)
$$

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 $\mathcal{O}\downarrow\mathcal{O}$

 $I.$ Bound: $\| [b, Dy]$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \lesssim \|b\|_{BMO(\nu)}$

$$
[b, \text{III}]f = \underbrace{(\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f}_{\text{V}} + \underbrace{(\pi_{\text{III}f}b - \text{III}\pi_f b)}_{\text{C}}
$$

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 $\mathcal{O}\downarrow\mathcal{O}$

 $I.$ Bound: $\| [b, Dy]$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \lesssim \|b\|_{BMO(\nu)}$

$$
[b, \text{III}]f = \underbrace{(\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f}_{\text{V}} + \underbrace{(\pi_{\text{III}f}b - \text{III}\pi_f b)}_{\text{Cov}}
$$

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 IIB Bound: $\|[b]$ Dyadic Shift] : $L^p(\mu) \to L^p(\lambda)$ $\|\leq \|b\|_{BMO(\nu)}$

$$
[b, \text{III}]f = \underbrace{(\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f}_{\text{V}} + \underbrace{(\pi_{\text{III}f}b - \text{III}\pi_f b)}_{\text{C}}.
$$

Known: $III : L^p(w) \rightarrow L^p(w)$

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 $I.$ Bound: $\| [b, Dy]$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \lesssim \|b\|_{BMO(\nu)}$

$$
[b, \text{III}]f = \underbrace{(\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f}_{\text{V}} + \underbrace{(\pi_{\text{III}f}b - \text{III}\pi_f b)}_{\text{V}}
$$

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 $\mathsf{Known}\colon\amalg:\mathit{L}^p(w)\to\mathit{L}^p(w)$

$$
\left|\frac{1}{\mu}\right|^{L^{p}(\mu)}
$$

 $I.$ Bound: $\| [b, Dy]$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \lesssim \|b\|_{BMO(\nu)}$

$$
[b, \text{III}]f = \underbrace{(\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f}_{\text{V}} + \underbrace{(\pi_{\text{III}f}b - \text{III}\pi_f b)}_{\text{V}}
$$

Known: $III : L^p(w) \rightarrow L^p(w)$

 $I.$ Bound: $\| [b, Dy]$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \lesssim \|b\|_{BMO(\nu)}$

$$
[b, \text{III}]f = (\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f + (\pi_{\text{III}f}b - \text{III}\pi_f b)
$$

Known: III : $L^p(w) \to L^p(w)$

 $I.$ Bound: $\| [b, Dy]$ adic Shift] : $L^p(\mu) \to L^p(\lambda) \| \lesssim \|b\|_{BMO(\nu)}$

$$
[b, \text{III}]f = (\pi_b \text{III} + \pi_b^* \text{III} - \text{III}\pi_b - \text{III}\pi_b^*)f + (\pi_{\text{III}f}b - \text{III}\pi_f b)
$$

Known: \text{III}: $L^p(w) \to L^p(w)$

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 \blacktriangleright Reduce to one-weight maximal and square function estimates!

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- \blacktriangleright Reduce to one-weight maximal and square function estimates!
- \blacktriangleright Key idea for this:

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- Reduce to one-weight maximal and square function estimates!
- \blacktriangleright Key idea for this: a weighted dyadic form of H^1 *BMO* duality

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- Reduce to one-weight maximal and square function estimates!
- \blacktriangleright Key idea for this: a weighted dyadic form of H^1 *BMO* duality (very nice for A₂ weights in particular)

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- Reduce to one-weight maximal and square function estimates!
- \blacktriangleright Key idea for this: a weighted dyadic form of H^1 *BMO* duality (very nice for A₂ weights in particular)
- \blacktriangleright $\nu = \mu^{1/p} \lambda^{1/p} \in A_2$!!!

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Outline

Introduction

Bloom's Result

Main Results

Upper Bound

Lower Bound: Key Idea

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$$
\|b\|_{BMO(\nu)}\lesssim \sum_{j=1}^n \|[b,R_j]:L^p(\mu)\rightarrow L^p(\lambda)\|.
$$

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$$
||b||_{BMO(\nu)} \lesssim \sum_{j=1}^n ||[b,R_j] : L^p(\mu) \to L^p(\lambda)||.
$$

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Follows the same strategy in CRW.

$$
||b||_{BMO(\nu)} \lesssim \sum_{j=1}^n ||[b,R_j] : L^p(\mu) \to L^p(\lambda)||.
$$

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Follows the same strategy in CRW. Key fact: equivalent definitions of Bloom BMO:

$$
||b||_{BMO(\nu)} \lesssim \sum_{j=1}^n \|[b,R_j]:L^p(\mu)\rightarrow L^p(\lambda)]|.
$$

Follows the same strategy in CRW. Key fact: equivalent definitions of Bloom BMO:

$$
||b||_{BMO(\nu)} := \sup_{Q} \frac{1}{\nu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q} | dx
$$

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Lower Bound

$$
||b||_{BMO(\nu)} \lesssim \sum_{j=1}^n \|[b,R_j]:L^p(\mu)\rightarrow L^p(\lambda)]|.
$$

Follows the same strategy in CRW. Key fact: equivalent definitions of Bloom BMO:

$$
||b||_{BMO(\nu)} := \sup_{Q} \frac{1}{\nu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q} | dx
$$

$$
||b||_{BMO (v)} \cong \sup_{Q} \left(\frac{1}{\mu(Q)} \int_{Q} \left| b(x) - \langle b \rangle_{Q} \right|^{p} d\lambda \right)^{1/p}
$$

Lower Bound

$$
||b||_{BMO(\nu)} \lesssim \sum_{j=1}^n \|[b,R_j]:L^p(\mu)\rightarrow L^p(\lambda)]|.
$$

Follows the same strategy in CRW. Key fact: equivalent definitions of Bloom BMO:

$$
\|b\|_{BMO(\nu)} := \sup_{Q} \frac{1}{\nu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q} dx
$$
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$$
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$$
||b||_{BMO^{2}(\nu)} := \sup_{Q} \left(\frac{1}{\nu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q}|^{2} d\nu^{-1}\right)^{1/2}
$$
\n
$$
||b||_{BMO(\nu)} \approx \sup_{Q} \left(\frac{1}{\mu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q}|^{p} d\lambda\right)^{1/p}
$$

Lower Bound

$$
||b||_{BMO(\nu)} \lesssim \sum_{j=1}^n \|[b,R_j]:L^p(\mu)\to L^p(\lambda)]|.
$$

Follows the same strategy in CRW. Key fact: equivalent definitions of Bloom BMO:

$$
||b||_{BMO(v)} := \sup_{Q} \frac{1}{v(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q} dx
$$

\n
$$
\cong \text{Muckenhoupt & Wheeler} \text{ (75)}
$$

\n
$$
||b||_{BMO^{2}(v)} := \sup_{Q} \left(\frac{1}{v(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q}|^{2} dv^{-1}\right)^{1/2}
$$

\n
$$
||b||_{BMO(v)} \cong \sup_{Q} \left(\frac{1}{\mu(Q)} \int_{Q} |b(x) - \langle b \rangle_{Q}|^{p} d\lambda\right)^{1/p}
$$

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