Svetlana Jitomirskaya

20-01-2017, MSRI

Sharp arithmetic spectral transitions and universal hierarchical structure of quasiperiodic eigenfunctions.

Consider the operator

$$(H\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + 2\lambda \cos 2\pi(\alpha n + \theta)\, \Psi_n$$

(λ ↑ corresponds to geometry of lattice; α ↑ magnetic flux)

Note that this is the simplest model that exhibits transitions as we change those parameters.

Fix α, λ. We define the level of exponent:

$$L(E) := \lim \frac{\int \ln \| A_n(\theta) \| \, d\theta}{n}$$

we have
$$\begin{pmatrix} \Psi_{n+1} \\ \Psi_n \end{pmatrix} = A_E(\theta + n\alpha) \begin{pmatrix} \Psi_n \\ \Psi_{n-1} \end{pmatrix}$$

where $A_E(\theta) = \begin{pmatrix} E - 2\lambda\cos(2\pi\theta) & -1 \\ 1 & 0 \end{pmatrix}$ is the transfer matrix.

$(H\Psi - E\Psi)$

Then,

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \prod_k A_E(\theta + k\alpha) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

$$= A_n(\theta) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

↑ n-step tranfer matrix

$L(E)$ can be computed explicitly

In fact,

$$L(E)\Big|_{E \in \sigma} = \max(0, \ln|\lambda|)$$ (proved by Bourgain, SJ 2002)

we have: $\lambda > 1 \Rightarrow L > 0$ on σ

$\lambda \leq 1 \Rightarrow L = 0$ on σ

Avila proved $\lambda < 1 \Rightarrow$ always absolutely continuous spectrum for all α, θ.

In this lecture, we will talk about: $\lambda > 1 \Rightarrow L > 0$ on σ

Since $L > 0$, there are no absolutely continuous spectrums

We consider two arithmetic parameters.

$$\beta(\alpha) := \overline{\lim} \frac{-\ln\|n\alpha\|}{n} \in [0, \infty] \quad \text{where } \|x\| = \text{dist to } \mathbb{Z}$$

$$= \overline{\lim} \frac{\ln q_{n+1}}{q_n} \quad \text{where } \frac{p_n}{q_n} \sim \alpha$$

and

$$\delta(\alpha, \theta) := \overline{\lim} \frac{-\ln\|2\theta + n\alpha\|}{n} \in [0, \infty]$$

We say α is Diophantine if $\beta(\alpha) = 0$

and θ is α-Diophantine if $\delta(\alpha) = 0$.

<u>Conjecture (94)</u>:

<u>part I</u>: Suppose θ is α-Diophantine

I_p: $L > \beta \Rightarrow$ pp spectrum

I_s: $L < \beta \Rightarrow$ sc spectrum

(pp = pure point, sc = singular continuous)

(transition happens at β)

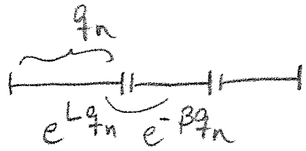
<u>part II</u>: Suppose α is Diophantine

II_p: $L > \delta \Rightarrow$ pp

II_s: $L < \delta \Rightarrow$ sc

<u>Question</u>: What is the reason for such conjecture?

Say we have 3 almost repeating pieces.

<u>if $\beta(\alpha) > 0$</u>: 

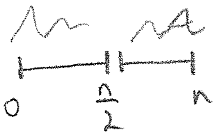
$|q_n\alpha - p_n| \sim \frac{1}{q_{n+1}}$

the eigenfunctions cannot decay as a corresponding local scale if the scale is large enough

If $q_{n+1} \sim e^{\beta q_n}$ then we have repetition

The decay is governed by the level of exponent

If $\delta > 0$:



this implies that $\frac{n}{2}$ will be almost a point of reflection.

(almost → precision governed by $e^{-\delta n}$)

$\Rightarrow$ the potential is reflected (almost)

Decay is at rate e^{-Ln}.

The following is a Joint work with Wencai Liu:

Theorem: In I_p and II_p, there exist explicit universal $f(k)$, $g(k)$ such that for any generalized eigenvalue E, for large K, we have

1) $$\frac{\ln \| U(k) \|}{k} \sim f(k)$$

2) $$\frac{\ln \| A_{K,E}(\theta) \|}{K} \sim g(k)$$

where $$U(k) = \begin{pmatrix} \Psi(k+1) \\ \Psi(k) \end{pmatrix} \qquad (H\Psi = E\Psi)$$

Note that in I_p, everything depends on α only (we have asymptotic behavior because of large k)

Moreover, we have for I_p : $L - \beta < f(k) < L$

II_p : $L - \delta < f(k) < L$

Some history about the conjecture:

<u>part I</u>:

$$L > \frac{16}{9}\beta \quad (\text{Avila - SJ})$$

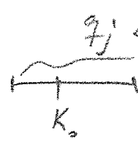
$$L < \frac{3}{2}\beta \quad (\text{Liu - Yuan})$$

Avila - You - Zhou proved $L > \beta$ for a.e. θ

I_s was also proved by Avila - You - Zhou.

<u>part II</u> was proved by Liu, SJ.

We say K_0 is a j-maximum if K_0 is a maximum on I where $|I| \sim q_j$



q_j ← sequence of denominators.

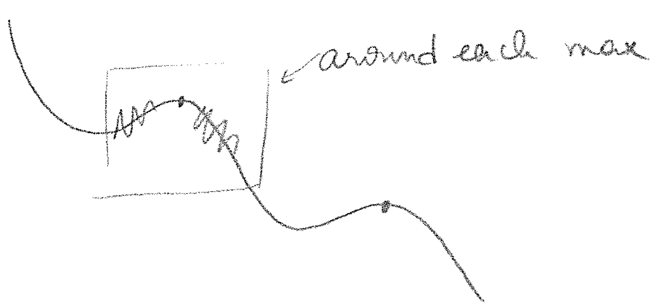
We say K_0 is a non-resonant j-max if

$$\|2\theta + (2K_0 + K)\alpha\| > \frac{c}{K^{\tau}} \quad \text{for } |K| \leq 2q_j$$

<u>Theorem</u>: Let K_0 be a non-resonant j-max

then

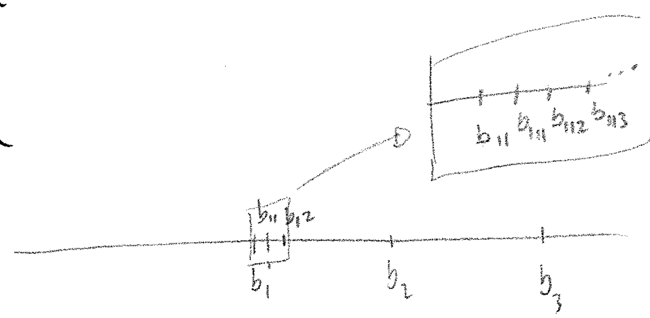
$$\frac{\|u(K_0+K)\|}{\|u(K_0)\|} \sim f(k) \quad \text{for } |k| < q_j$$



We have a hierarchical structure of local maxima $b_{a_j \cdots a_{j-s}}$ which satisfy the following.

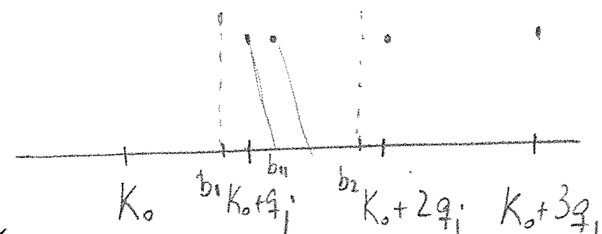
Let K_0 be a global maximum

$\exists\, n_0(\alpha, \varepsilon)$ such that



1) $b_{a_j \cdots a_{j-s}}$ is $(j-s)$ - maximum

2) $|b_{a_j} - K_0 - a_j q_j| < q_{n_0}$



3) $|b_{a_j a_{j-1}} - b_{a_j} - a_{j-1} q_{j-1}| < q_{n_0}$

$\vdots$

$$|b_{a_j \cdots a_{j-k}} - b_{a_j \cdots a_{j-k+1}} - a_{j-k} q_{j-k}| < q_{n_0+k}$$

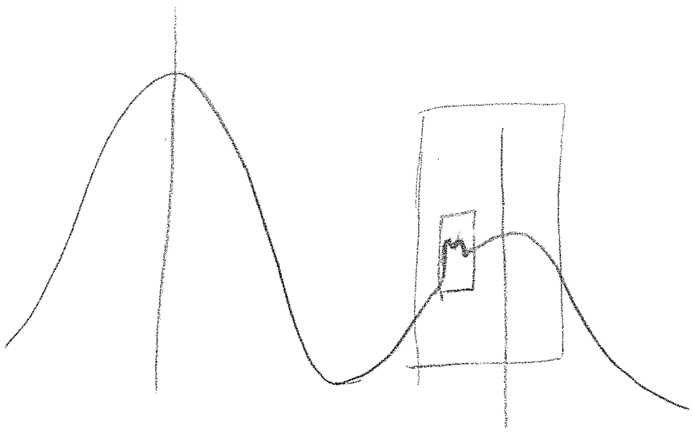
and we have,

$$\frac{\| u(b_{a_j \cdots a_{j-k}} + x) \|}{\| u(b_{a_j \cdots a_{j-k}}) \|} \sim f(x) \quad \text{for } |x| < c\, q_{j-k}$$

For $\mathbb{I}_p$, there is a hierarchical structure of the form $b_{k_j k_{j-1} \cdots k_{j-s}}$ so that

$$\frac{\| u(b_{k_j \cdots k_{j-s}} + x) \|}{\| u(b_{k_j \cdots k_{j-s}}) \|} \sim f((-1)^s x)$$

Note:
If $\delta > 0$ then we have infinitely many almost reflections at $\frac{n}{2}$.



Lastly, note that f is the behavior of eigenfunctions and g is the behavior of the norm of the transfer matrix.

Question: why isn't this redundant?

we have

$\| A_{n,z}(\theta) \| \sim g(n)$: norm of most expanded vector

$\| u_n \| \sim f(n)$: norm of most contracted vector

Let $\gamma(n) =$ angle between $(u, \bar{u})$ ← u: most contracted, $\bar{u}$: most expanded

we have

$$\overline{\lim} \frac{-\ln \gamma(n)}{n} = \beta_\delta .$$

Quasiperiodic Schrodinger operators: sharp arithmetic spectral transitions and universal hierarchical structure of eigenfunctions

S. Jitomirskaya

UCI

MSRI, January 20, 2017

Almost Mathieu operators

$$(H_{\lambda,\alpha,\theta}\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + \lambda v(\theta + n\alpha)\Psi_n$$

$v(\theta) = 2\cos 2\pi(\theta)$, α irrational,

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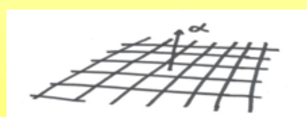
Tight-binding model of 2D Bloch electrons in magnetic fields



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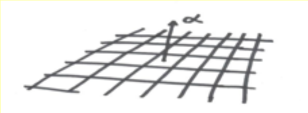

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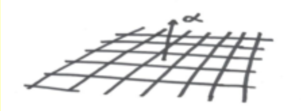

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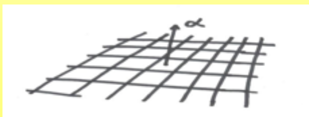

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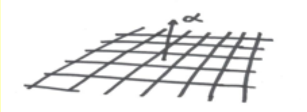

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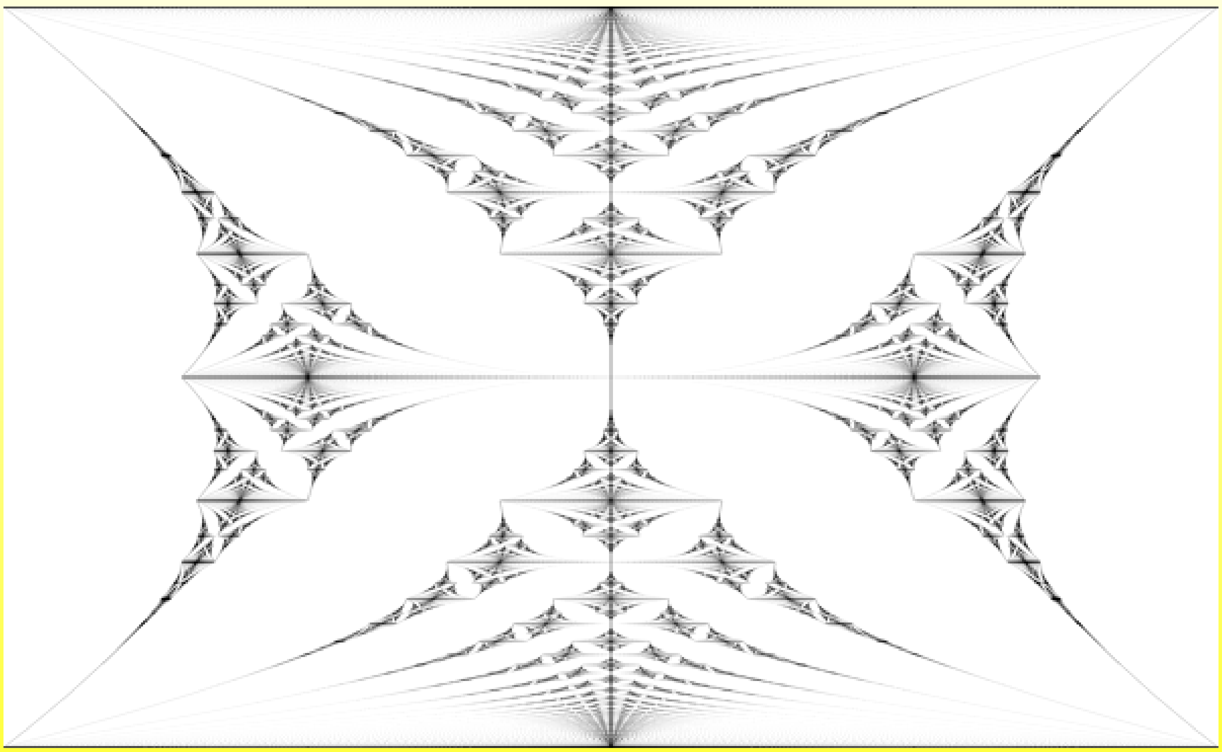
- Is called Harper's model
- With a choice of Landau gauge effectively reduces to h_θ
- α is a dimensionless parameter equal to the ratio of flux through a lattice cell to one flux quantum.

Hofstadter butterfly



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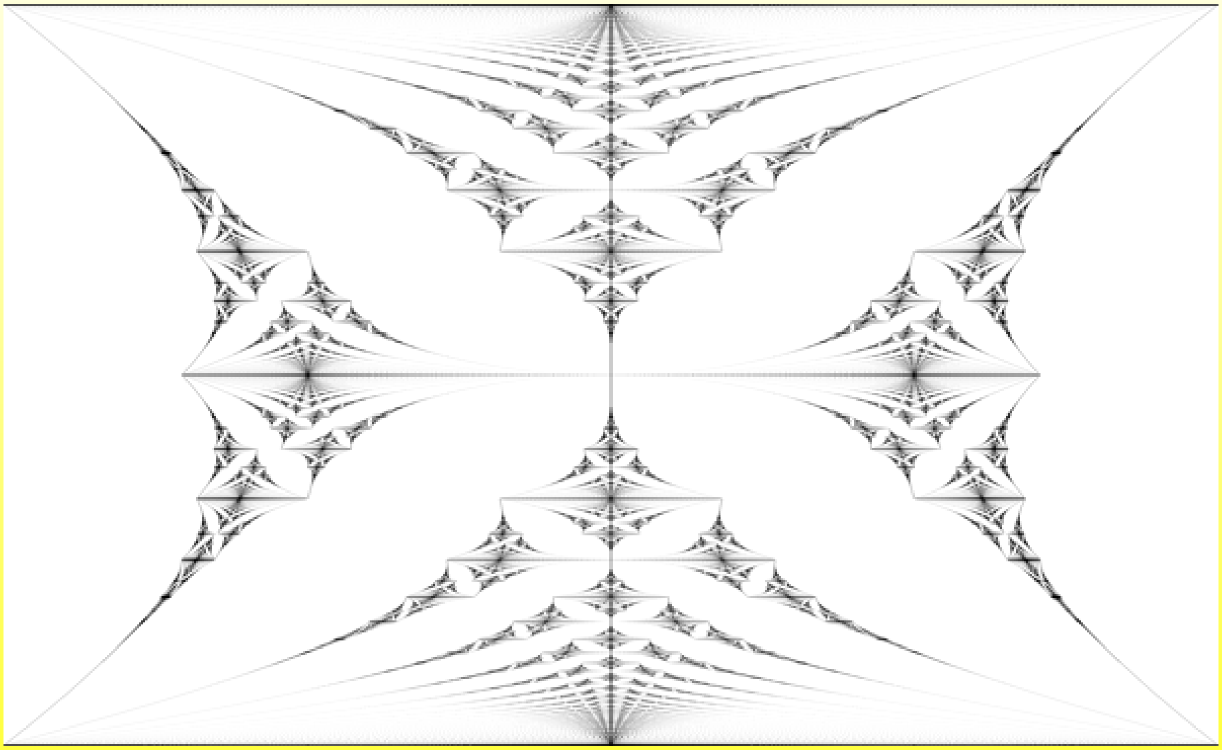

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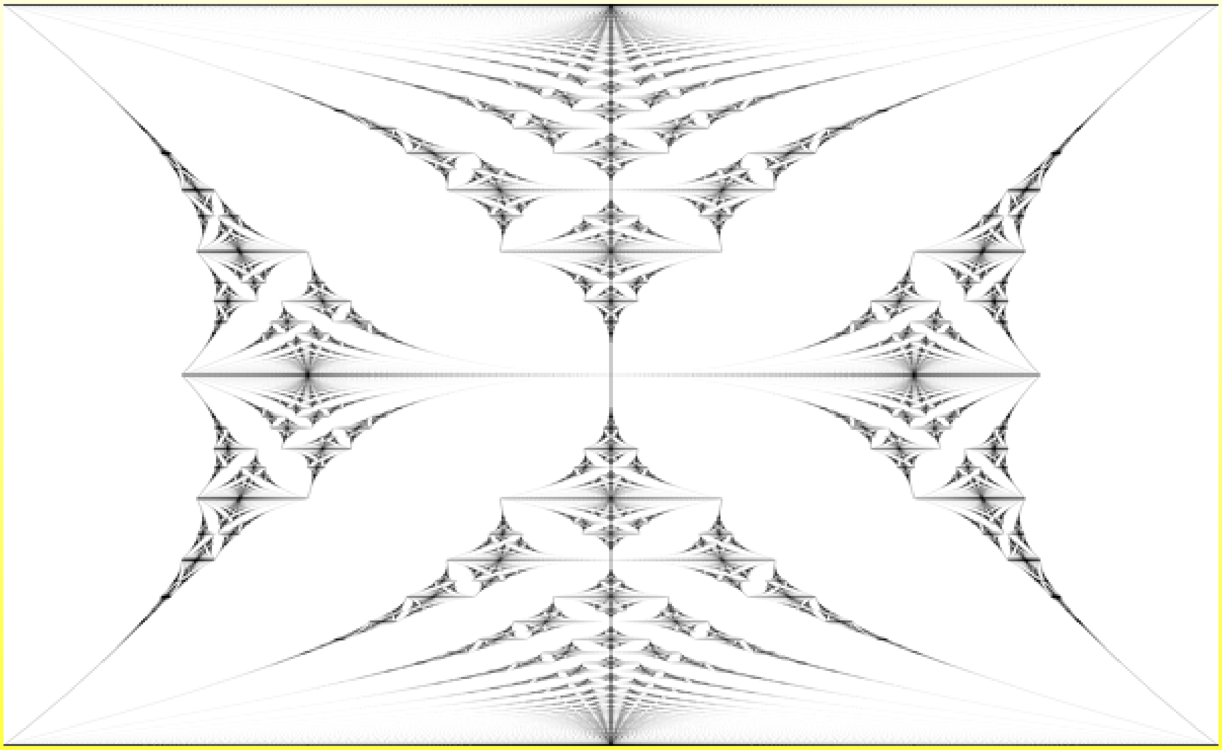

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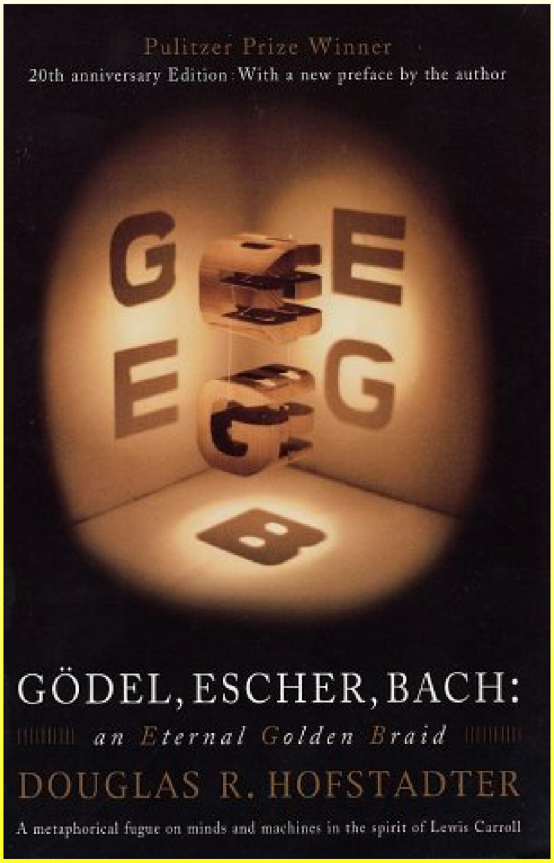
David Jennings described it as a picture of God

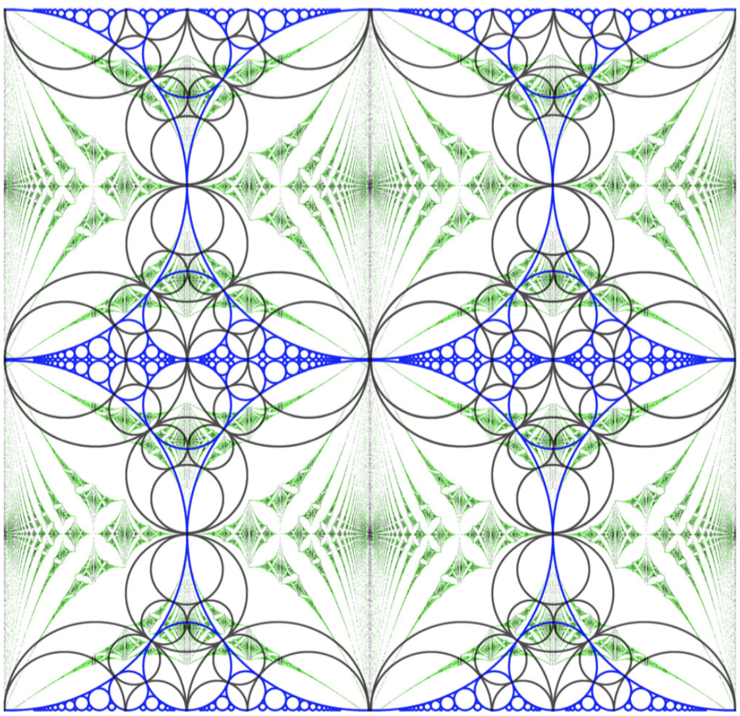

Pulitzer Prize Winner
20th anniversary Edition : With a new preface by the author
GÖDEL, ESCHER, BACH:
an Eternal Golden Braid
DOUGLAS R. HOFSTADTER
A metaphorical fugue on minds and machines in the spirit of Lewis Carroll

Butterfly in the Quantum World

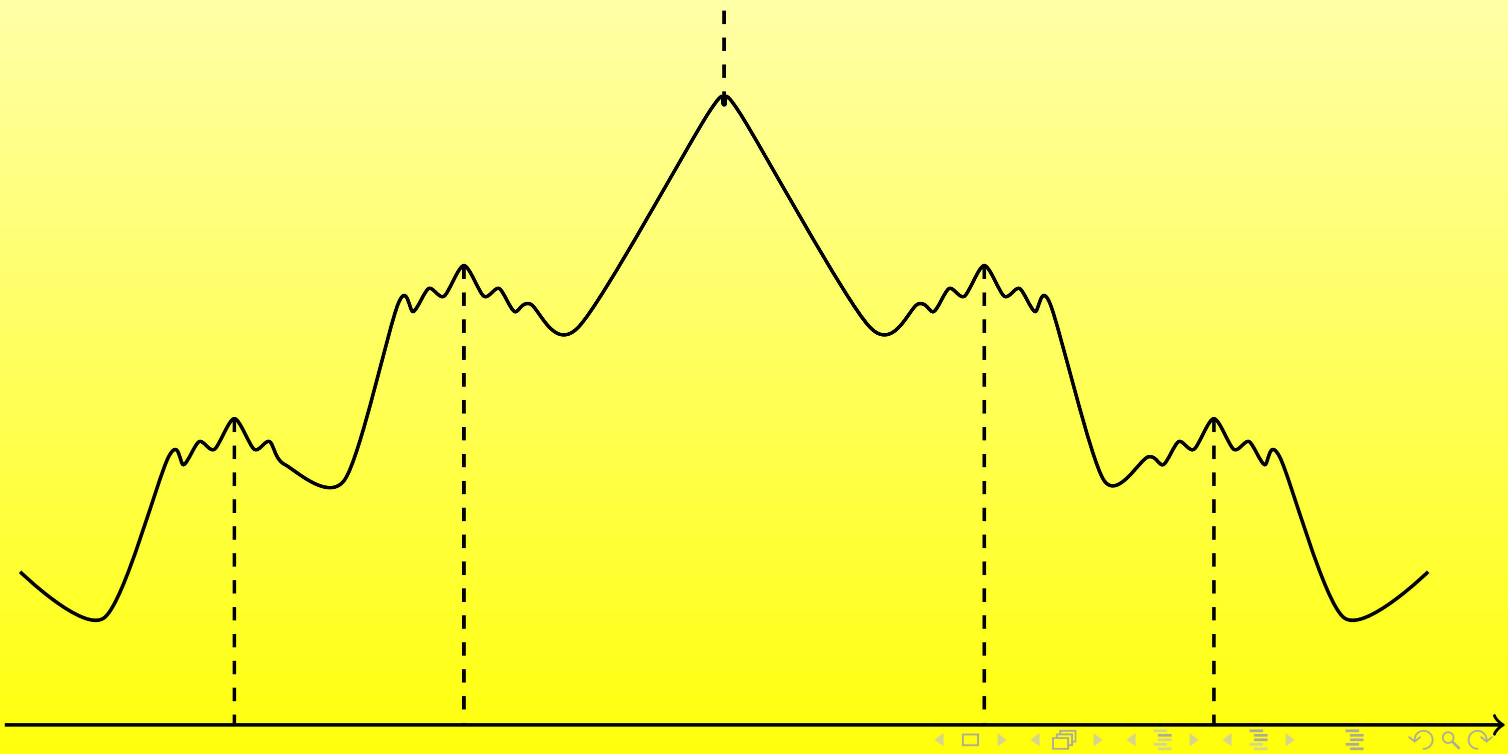
The story of the most fascinating quantum fractal

Indubala I Satija

with contributions by Douglas Hofstadter



Hierarchical structure driven by the continued fraction expansion of the magnetic flux: eigenfunctions



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Today: universal self-similar exponential structure of eigenfunctions throughout the entire localization regime.

Arithmetic spectral transitions

1D Quasiperiodic operators:

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- originally approached by KAM (Dinaburg, Sinai, Bellissard, Frohlich-Spencer-Wittwer, Eliasson)
- nonperturbative methods (SJ, Bourgain-Goldstein for $L > 0$; Last,SJ,Avila for $L = 0$) reduced the transition to the transition in the Lyapunov exponent (for analytic v):
  $L(E) > 0$ implies pp spectrum for a.e. α, θ
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$$L(\alpha, E) \quad := \quad \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log ||A^E_{(n)}(x)|| dx \ ,$$

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Arithmetic transitions in the supercritical ($L > 0$) regime

Small denominators - resonances - $(v(\theta + k\alpha) - v(\theta + \ell\alpha))^{-1}$ are in competition with $e^{L(E)|\ell - k|}$.
L very large compared to the resonance strength leads to more localization
L small compared to the resonance strength leads to delocalization

Pure point to singular continuous transition conjecture

Exponential strength of a resonance:

$$\beta(\alpha) := \limsup_{n\to\infty} -\frac{\ln ||n\alpha||_{\mathbb{R}/\mathbb{Z}}}{|n|}$$

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$\lambda > 1 \rightarrow$ no ac spectrum (Ishii-Kotani-Pastur)

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Conjecture for the sharp transition (1994):

- If $\beta(\alpha) = 0$, then $\lambda_0 = e^{\delta(\alpha,\theta)}$ is the transition line:
  - $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for $|\lambda| < e^{\delta(\alpha,\theta)}$,
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sc spectrum for $\beta = \infty$ proved in Gordon, Avron-Simon (82), and for $\delta = \infty$ in SJ-Simon (94)

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Conjecture for the sharp transition (1994):

- If $\beta(\alpha) = 0$, then $\lambda_0 = e^{\delta(\alpha,\theta)}$ is the transition line:
  - $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for $|\lambda| < e^{\delta(\alpha,\theta)}$,
  - $H_{\lambda,\alpha,\theta}$ has Anderson localization (stronger than pure point spectrum) for $|\lambda| > e^{\delta(\alpha,\theta)}$.

  (all θ, Diophantine α)
- If $\delta(\alpha, \theta) = 0$, then $L(E) = \beta(\alpha)$ is the transition line.
  - $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for $L(E) < \beta(\alpha)$
  - $H_{\lambda,\alpha,\theta}$ has Anderson localization for $L(E) > \beta(\alpha)$.

  (all α, Diophantine θ)

sc spectrum for $\beta = \infty$ proved in Gordon, Avron-Simon (82), and for $\delta = \infty$ in SJ-Simon (94)
pp spectrum for $\beta = \delta = 0$ proved in SJ (99).

Asymptotics in the pp regime

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$$f(|k|)e^{-\varepsilon|k|} \leq ||U(k)|| \leq f(|k|)e^{\varepsilon|k|},$$

and

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Let $\frac{p_n}{q_n}$ be the continued fraction expansion of α. For any $\frac{q_n}{2} \leq k < \frac{q_{n+1}}{2}$, define explicit functions $f(k), g(k)$ as follows(depend on α through the sequence of q_n):

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$$f(k) = e^{-|k-\ell q_n| \ln|\lambda|} \bar{r}_\ell^n + e^{-|k-(\ell+1)q_n| \ln|\lambda|} \bar{r}_{\ell+1}^n,$$

and

$$g(k) = e^{-|k-\ell q_n| \ln|\lambda|} \frac{q_{n+1}}{\bar{r}_\ell^n} + e^{-|k-(\ell+1)q_n| \ln|\lambda|} \frac{q_{n+1}}{\bar{r}_{\ell+1}^n},$$

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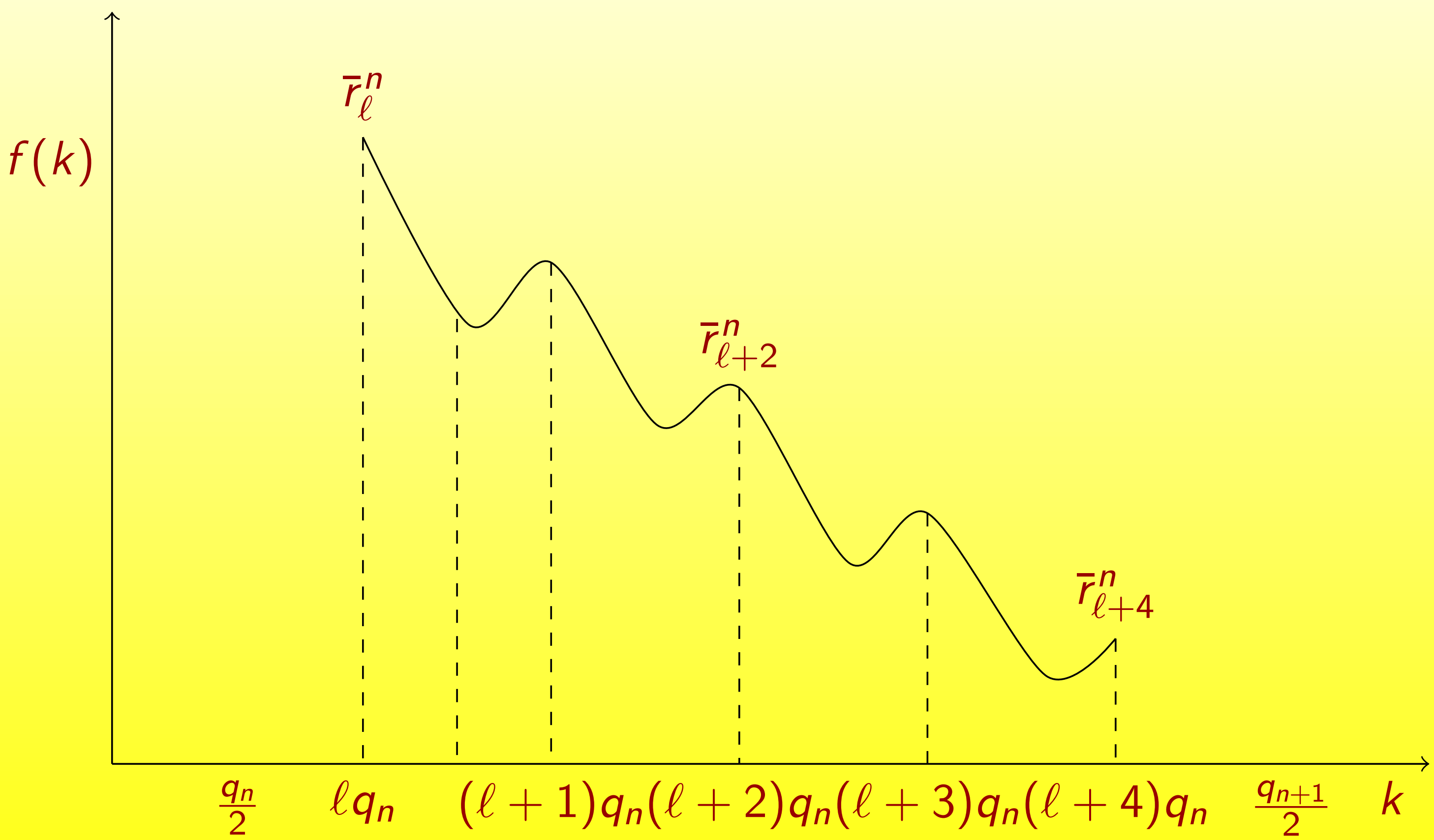
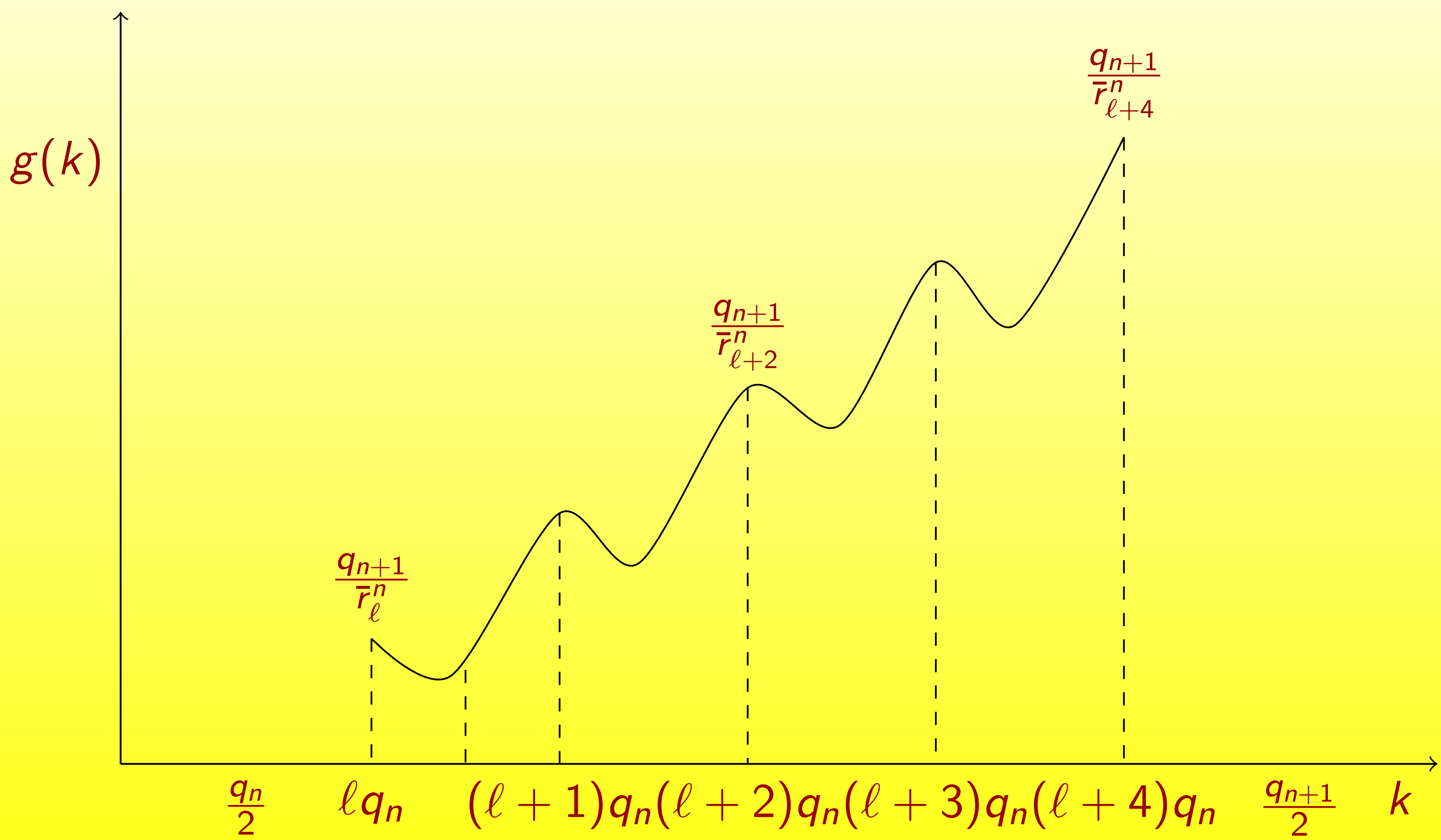
Note: $f(k)$ decays exponentially and $g(k)$ grows exponentially. However the decay rate and growth rate are not always the same.

The behavior of $f(k)$



The behavior of $g(k)$



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Corollary

The arithmetic spectral transition conjecture holds as stated.

History

Localization Method:

- Avila-SJ: if $|\lambda| > e^{\frac{16}{9}\beta(\alpha)}$ and $\delta(\alpha,\theta) = 0$, then $H_{\lambda,\alpha,\theta}$ satisfies AL (Ten Martini Problem)

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Local j-maxima

Local j-maximum is a local maximum on a segment $|I| \sim q_j$.
A local j-maximum k_0 is *nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{{q_{j-1}}^{\nu}},$$

for all $|k| \leq 2q_{j-1}$ and

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \qquad (0.1)$$

for all $2q_{j-1} < |k| \leq 2q_j$.
A local j-maximum is *strongly nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \qquad (0.2)$$

for all $0 < |k| \leq 2q_j$.

Universality of behavior at all (strongly) nonresonant local maxima:

Theorem

(SJ-W.Liu, 16) Suppose k_0 is a local j-maximum. If k_0 is nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \qquad (0.3)$$

for all $2s \in I$, $|s| > \frac{q_{j-1}}{2}$.
If k_0 is strongly nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \qquad (0.4)$$

for all $2s \in I$.

Universal hierarchical structure

All α, Diophantine θ, pp regime. Let k_0 be the global maximum

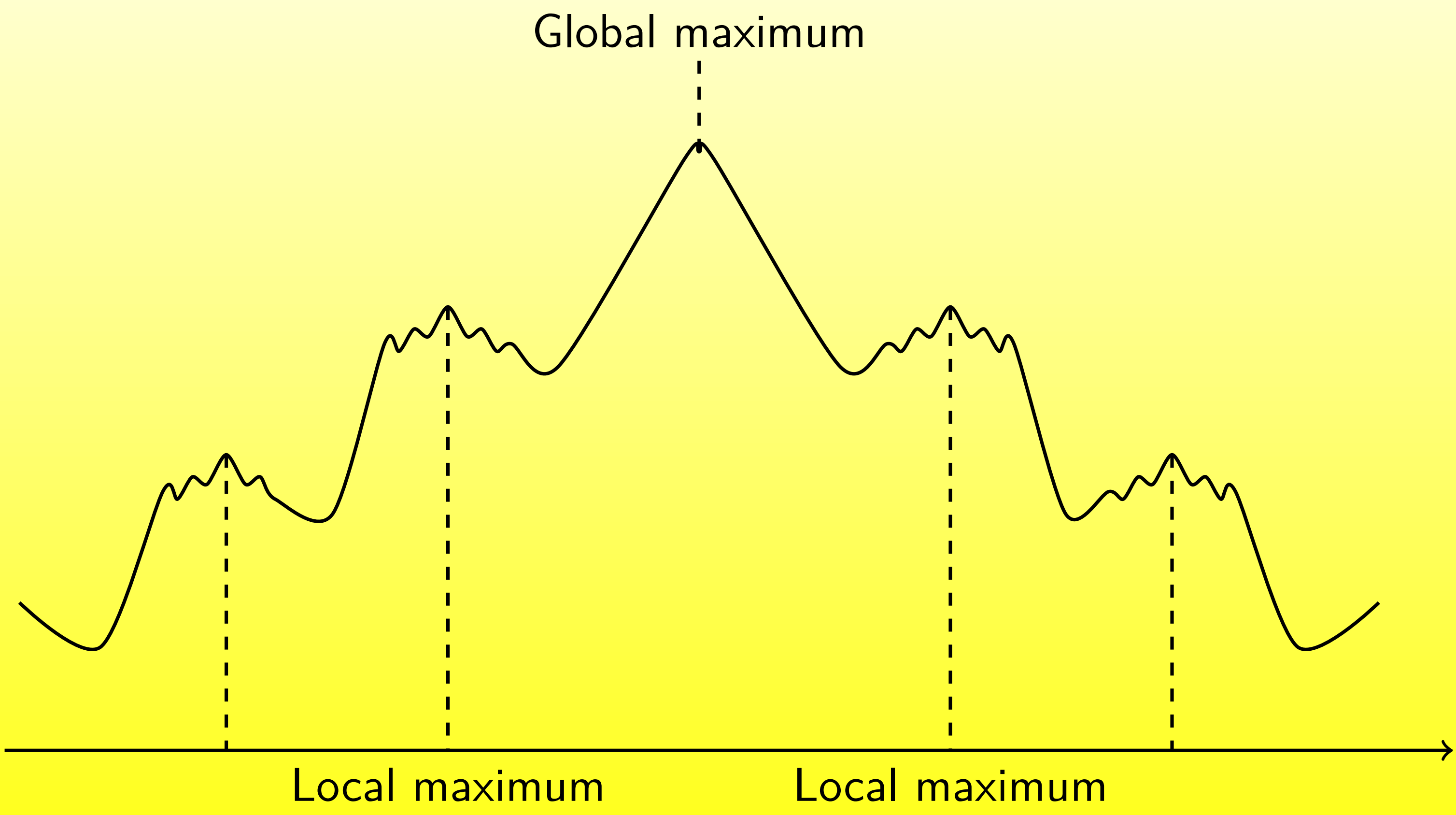
Theorem

(SJ-W. Liu, 16) There exists $\hat{n}_0(\alpha, \lambda, \varsigma, \epsilon) < \infty$ *such that for any* $k \geq \hat{n}_0$, $n_{j-k} \geq \hat{n}_0 + k$, *and* $0 < a_{n_i} < e^{\varsigma \ln|\lambda| q_{n_i}}$, $i = j-k, \ldots, j$, *for all* $0 \leq s \leq k$ *there exists a local* n_{j-s}*-maximum* $b_{a_{n_j}, a_{n_{j-1}}, \ldots, a_{n_{j-s}}}$ *such that the following holds:*
I. $|b_{a_{n_j}} - (k_0 + a_{n_j} q_{n_j})| \leq q_{\hat{n}_0+1}$,
II.For $s \leq k$, $|b_{a_{n_j}, \ldots, a_{n_{j-s}}} - (b_{a_{n_j}, \ldots, a_{n_{j-s+1}}} + a_{n_{j-s}} q_{n_{j-s}})| \leq q_{\hat{n}_0+s+1}$.
III. if $q_{\hat{n}_0+k} \leq |(x - b_{a_{n_j}, a_{n_{j-1}}, \ldots, a_{n_{j-k}}}| \leq c q_{n_{j-k}}$, *then for* $s = 0, 1, \ldots, k$,

$$f(x_s) e^{-\varepsilon|x_s|} \leq \frac{||U(x)||}{||U(b_{a_{n_j}, a_{n_{j-1}}, \ldots, a_{n_{j-s}}})||} \leq f(x_s) e^{\varepsilon|x_s|},$$

Moreover, every local n_{j-s}*-maximum on the interval* $b_{a_{n_j}, a_{n_{j-1}}, \ldots, a_{n_{j-s+1}}} + [-e^{\epsilon \ln \lambda q_{n_{j-s}}}, e^{\epsilon \ln \lambda q_{n_{j-s}}}]$ *is of the form* $b_{a_{n_j}, a_{n_{j-1}}, \ldots, a_{n_{j-s}}}$ *for some* $a_{n_{j-s}}$.

Universal hierarchical structure of the eigenfunctions



Universal reflexive-hierarchical structure

Theorem

(SJ-W. Liu,16) For Diophantine α and all θ in the pure point regime there exists a hierarchical structure of local maxima as above, such that

$$f((-1)^{s+1}x_s)e^{-\varepsilon|x_s|} \leq \frac{||U(x)||}{||U(b_{K_j,K_{j-1},\ldots,K_{j-s}})||} \leq f((-1)^{s+1}x_s)e^{\varepsilon|x_s|},$$

where $x_s = x - b_{K_j,K_{j-1},\ldots,K_{j-s}}$.

Further corollaries

Corollary

Let $\psi(k)$ *be any solution to* $H_{\lambda,\alpha,\theta}\psi = E\psi$ *that is linearly independent with respect to* $\phi(k)$. *Let* $\bar{U}(k) = \begin{pmatrix} \psi(k) \\ \psi(k-1) \end{pmatrix}$, *then*

$$g(|k|)e^{-\varepsilon|k|} \leq ||\bar{U}(k)|| \leq g(|k|)e^{\varepsilon|k|}.$$

Let $0 \leq \delta_k \leq \frac{\pi}{2}$ be the angle between vectors $U(k)$ and $\bar{U}(k)$.

Corollary

We have

$$\limsup_{k\to\infty} \frac{\ln \delta_k}{k} = 0,$$

and

$$\liminf_{k\to\infty} \frac{\ln \delta_k}{k} = -\beta.$$

Corollary

We have

i)
$$\limsup_{k\to\infty} \frac{\ln ||A_k||}{k} = \limsup_{k\to\infty} \frac{\ln ||\bar{U}(k)||}{k} = \ln |\lambda|,$$

ii)
$$\liminf_{k\to\infty} \frac{\ln ||A_k||}{k} = \liminf_{k\to\infty} \frac{\ln ||\bar{U}(k)||}{k} = \ln |\lambda| - \beta.$$

iii) *Outside an explicit sequence of lower density zero,*
$$\lim_{k\to\infty} \frac{\ln ||A_k||}{k} = \lim_{k\to\infty} \frac{\ln ||\bar{U}(k)||}{k} = \ln |\lambda|.$$

Corollary

We have

i) $\limsup_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln|\lambda|,$

ii) $\liminf_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln|\lambda| - \beta.$

iii) *There is an explicit sequence of upper density* $1 - \frac{1}{2}\frac{\beta}{\ln|\lambda|}$, *along which*

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iv) *There is an explicit sequence of upper density* $\frac{1}{2}\frac{\beta}{\ln|\lambda|}$,*along which*

$$\limsup_{k\to\infty} \frac{-\ln ||U(k)||}{k} < \ln|\lambda|.$$

Further applications

- Upper bounds on fractal dimensions of spectral measures and quantum dynamics for trigonometric polynomials (SJ-W.Liu-S.Tcheremchantzev, SJ-W.Liu).
- The exact rate for exponential dynamical localization in expectation for the Diophantine case (SJ-H.Krüger-W.Liu). The first result of its kind, for any model.
- The same universal asymptotics of eigenfunctions for the Maryland Model (R. Han-SJ-F.Yang).

Key ideas of the proof

Resonant points (small divisors): $k : ||k\alpha||_{\mathbb{R}/\mathbb{Z}}$ or $||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}$ is small.

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- New way to deal with resonant points in the positive Lyapunov regime (supercritical regime)

Key ideas of the proof

Resonant points (small divisors): $k : ||k\alpha||_{\mathbb{R}/\mathbb{Z}}$ or $||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}$ is small.

- New way to deal with resonant points in the positive Lyapunov regime (supercritical regime)
- Develop Gordon and palindromic methods to study the trace of transfer matrices to obtain lower bounds on solutions
  Gordon potential (periodicity): $|V(j+q_n) - V(j)|$ is small (control by $||q_n\alpha|| \simeq e^{-\beta(\alpha)q_n}$)
  palindromic potential (symmetry): $|V(k-j) - V(j)|$ is small (control by $||2\theta + k\alpha|| \simeq e^{-\delta(\alpha,\theta)|k|}$)
- Bootstrap starting around the (local) maxima leads to effective estimates
- Reverse induction proof that local $j-1$-maxima are close to aq_{j-1} shifts of the local j-maxima, up to a constant scale
- Deduce that all the local maxima are (strongly) non-resonant and apply reverse induction

Assume E is a generalized eigenvalue and ϕ is the associated generalized eigenfunction ($|\phi(n)| < 1 + |n|$). Let φ be another solution of $Hu = Eu$. Let $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$ and $\bar{U}(k) = \begin{pmatrix} \varphi(k) \\ \varphi(k-1) \end{pmatrix}$.

Step 1: Sharp estimates for the non-resonant points.

- $||U(k)|| \simeq e^{-\ln\lambda|k-k_i|}||U(k_i)|| + e^{-\ln\lambda|k-k_{i+1}|}||U(k_{i+1})||$
- $||\bar{U}(k)|| \simeq e^{-\ln\lambda|k-k_i|}||\bar{U}(k_i)|| + e^{-\ln\lambda|k-k_{i+1}|}||\bar{U}(k_{i+1})||$

where k_i is the resonant point and $k \in [k_i, k_{i+1}]$.

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Step 2: Sharp estimates for the resonant points.

- $||U(k_{i+1})|| \simeq e^{-c(k_i,k_{i+1})|k_{i+1}-k_i|}||U(k_i)||$
- $||\bar{U}(k_{i+1})|| \simeq e^{c'(k_i,k_{i+1})|k_{i+1}-k_i|}||\bar{U}(k_i)||$

where $c(k_i, k_{i+1}), c'(k_i, k_{i+1})$ can be given explicitly.

Current quasiperiodic preprints

Almost Mathieu operator:

- Avila-You-Zhou: sharp transition in α between pp and sc
- Avila-You-Zhou: dry Ten Martini, non-critical, all α
- Shamis-Last, Krasovsky, SJ- S. Zhang: gap size/dimension results for the critical case
- Avila-SJ-Zhou: critical line $\lambda = e^{\beta}$
- Damanik-Goldstein-Schlag-Voda: homogeneous spectrum, Diophantine α
- W. Liu-SJ: sharp transitions in α and θ and universal (reflective) hierarchical structure

Unitary almost Mathieu:

Fillman-Ong-Z. Zhang: complete a.e. spectral description

Current quasiperiodic preprints

Extended Harper's model:

- Avila-SJ-Marx: complete spectral description in the coupling phase space (+Erdos-Szekeres conjecture!)
- R. Han: an alternative argument
- R. Han-J: sharp transition in α between pp and sc spectrum in the positive Lyapunov exponent regime
- R. Han: dry Ten Martini (non-critical Diophantine)

General 1-frequency quasiperiodic:

analytic: SJ- S. Zhang: sharp arithmetic criterion for full spectral dimensionality (quasiballistic motion)

R. Han-SJ: sharp topological criterion for dual reducibility to imply localization

Damanik-Goldstein-Schlag-Voda: homogeneous spectrum, supercritical

monotone: SJ-Kachkovskiy: *all* coupling localization

meromorphic: SJ-Yang: sharp criterion for sc spectrum

Current quasiperiodic preprints

Maryland model:

W. Liu-SJ: complete arithmetic spectral transitions for *all* λ, α, θ

W. Liu: surface Maryland model

SJ-Yang: a constructive proof of localization

General Multi-frequency:

- R. Han-SJ: localization-type results with arithmetic conditions (general zero entropy dynamics; including the skew shift)
- R. Han-Yang: generic continuous spectrum
- Hou-Wang-Zhou: ac spectrum for Liouville (presence)
- Avila-SJ: ac spectrum for Liouville (absence)

Deift's problem (almost periodicity of KdV solutions with almost periodic initial data) :

Binder-Damanik-Goldstein-Lukic: a solution under certain conditions.