Fundamental solutions and Green functions for non-homogeneous elliptic systems

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January 20, 2017

Fundamental Solutions

Let L denote some partial differential operator defined in $\mathbb{R}^n$.

Formally, a fundamental solution is some function (or distribution) $\Gamma(x,y)$, defined on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ for which

$$L_x \Gamma(x,y) = \delta_y(x) = \delta(x-y),$$

where δ denotes the Dirac delta function.

Green functions

Let $\Omega \subset \mathbb{R}^n$ be open and connected.

A Green function $G(x,y)$, defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$, satisfies

$$L_x G(x,y) = \delta_y(x) \quad \text{in } \Omega$$

and

$$G(x,y) = 0 \quad x \in \partial\Omega.$$

Why we like fundamental solutions

Suppose we want to solve

$$Lu = f \quad \text{in } \mathbb{R}^n.$$

If f is reasonable (e.g. $f \in C_c^\infty(\mathbb{R}^n)$), then we can use the fundamental solution and superposition to find the solution.

Let

$$u = \Gamma * f$$

That is,

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x, y) f(y)\, dy.$$

To establish that u is a solution, we need to justify differentiation under the integral sign.

An example

Consider $L = \Delta = \operatorname{div} \nabla$, the Laplacian, in $\mathbb{R}^n$ for some $n \geq 3$.

Since the Laplacian is radially symmetric, we look for $\Gamma(x,y) = F(r)$, where $r := |x - y|$.

If we take $\Gamma(x,y) = c_n r^{2-n}$, where c_n is some constant, then

$$\Delta_x \Gamma(x,y) = 0 \quad \text{in } \mathbb{R}^n \setminus \{y\}.$$

Moreover, $\Gamma(x,y)$ has a singularity at $y = x$.

Choosing the constant

How do we choose c_n?

Let $\varepsilon > 0$. Then

$$1 = \iint_{B_\varepsilon(y)} \delta_y(x)\, dx = \iint_{B_\varepsilon(y)} \Delta_x \Gamma(x,y)\, dx = \iint_{B_\varepsilon(y)} \Delta_x \left(c_n r^{2-n}\right) dx$$

$$= c_n \iint_{B_\varepsilon(y)} \nabla_x \cdot \nabla_x \left(r^{2-n}\right) dx = c_n \int_{\partial B_\varepsilon(y)} \frac{\partial \left(r^{2-n}\right)}{\partial r} dS$$

$$= c_n (2-n) \int_{\partial B_\varepsilon(y)} r^{1-n} dS = c_n (2-n) \left|S^{n-1}\right|$$

Thus,

$$c_n = \frac{1}{(2-n)\left|S^{n-1}\right|}.$$

Short answer: Use the divergence theorem.

More general settings

How do we find fundamental solutions for more general second-order elliptic partial differential operators?

e.g. Suppose $L = -\operatorname{div}(A\nabla)$.

An important tool is the Lax-Milgram Theorem.

Lax-Milgram Theorem

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $||\cdot||$.
Let $B : H \times H \to \mathbb{R}$ be a bounded, coercive bilinear mapping, i.e. there exists $\gamma, \Lambda > 0$ such that for any $u, v \in H$,

$$B[u,v] \leq \Lambda ||u|| \, ||v||$$

$$B[u,u] \geq \gamma ||u||^2 .$$

Let $f : H \to \mathbb{R}$ be a bounded, linear functional on H.
Then there exists a unique $u \in H$ such that

$$B[u,v] = \langle f, v \rangle$$

for every $v \in H$.

The connection to elliptic operators

Recall that $u \in W^{1,2}(\Omega)$ is a weak solution to $Lu = f$ in $\Omega \subset \mathbb{R}^n$ if for every $\phi \in W_0^{1,2}(\Omega)$, we have that

$$B[u, \phi] = \langle f, \phi \rangle$$

where $B[\cdot, \cdot]$ is the bilinear form naturally associated to L.

e.g. If $L = -\partial_i (a_{ij} \partial_j) = -\operatorname{div}(A \nabla)$, then

$$B[u, \phi] = \int a_{ij}(x) \partial_j u(x) \partial_i \phi(x) \, dx$$

Starting point

Fix $y \in \mathbb{R}^n$.

Under reasonable assumptions on ϕ,

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho|} \int_{B_\rho(y)} \phi(x)\, dx = \phi(y) = \int \phi(x)\, \delta_y(x)\, dx.$$

And for each $\rho > 0$, the mapping

$$W_0^{1,2}(\Omega) \ni \phi \mapsto \frac{1}{|B_\rho|} \int_{B_\rho(y)} \phi(x)\, dx$$

is a bounded linear functional.

Limiting

Assuming that we have an appropriate Hilbert space, this allows us to define $\Gamma_\rho(x,y)$, the solution to

$$B\left[\Gamma_\rho(\cdot,y),\phi\right] = \frac{1}{|B_\rho|}\int_{B_\rho(y)} \phi = \left\langle \frac{1}{|B_\rho|}\mathbb{1}_{B_\rho},\phi \right\rangle$$

Idea: Use a limiting procedure to derive $\Gamma(x,y)$ from the set $\left\{\Gamma_\rho(x,y)\right\}_{\rho>0}$.

*Boundedness/continuity of solutions can be used.

History

Grüter and Widman (1982) constructed Green functions for

$$L = -\partial_i \left(a_{ij} \partial_j \right) = -\operatorname{div} A \nabla$$

where A is elliptic and bounded

$$\lambda \left| \xi \right|^2 \leq a_{ij} \left(x \right) \xi_i \xi_j \leq \Lambda \left| \xi \right|^2$$

and $\Omega \subset \mathbb{R}^n$ is bounded.

Tools they used:

- Lax-Milgram theorem
- Local boundedness (aka Moser boundedness)
- Harnack inequality
- Maximum principle

History

The work of Grüter and Widman built on the results of Littman, Stampacchia, Weinberger (1963) who considered the same operators with a symmetry condition:

$$a_{ij} = a_{ji}$$

More recent history

Hofmann and Kim (2007) followed the general approach of Grüter and Widman and established existence, uniqueness and a priori estimates for

- fundamental solutions in $\mathbb{R}^n$, $n \geq 3$; and
- Green functions in arbitrary open, connected domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$

for *systems* of second-order, uniformly elliptic, divergence-form operators.

Elliptic systems

The operators studied by Hofmann and Kim are formally given by

$$Lu = -D_\alpha \left(\mathbf{A}^{\alpha\beta} D_\beta \mathbf{u} \right),$$

where $\mathbf{u} = \left(u^1, \ldots, u^N\right)$ for some $N \in \mathbb{N}$,

$$A_{ij}^{\alpha\beta}(x)\, \xi_\beta^j \xi_\alpha^i \geq \lambda \left|\xi\right|^2 := \lambda \sum_{i=1}^{N} \sum_{\alpha=1}^{n} \left|\xi_\alpha^i\right|^2$$

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \left| A_{ij}^{\alpha\beta}(x) \right|^2 \leq \Lambda^2,$$

for some $0 < \lambda, \Lambda < \infty$ and for all x in the domain.

Why do we care about systems?

Systems are a natural generalizations of the scalar setting.

Moreover, systems give us a framework in which we can analyze complex-valued equations.

Systems are hard to work with

In the systems setting, we can formulate the Lax-Milgram theorem.

However, there are a number of challenges because

- de Giorgi-Nash-Moser theory can fail
- No maximum principle
- Harnack inequality ??

Hofmann-Kim approach

To overcome the challenges listed above, Hofmann and Kim assumed that all solutions to

$$L\mathbf{u} = \mathbf{0}$$

satisfy a Hölder continuity condition of the following form:

$$\int_{B_r} |D\mathbf{u}|^2 \leq C \left(\frac{r}{R}\right)^{n-2+2\eta} \int_{B_R} |D\mathbf{u}|^2$$

where $\eta > 0$, $0 < r < R$.

Non-homogeneous systems

Can we establish existence, uniqueness and a priori estimates for fundamental solutions to non-homogenous elliptic operators?

Such operators are formally given by

$$\mathscr{L}\mathbf{u} := -D_\alpha\left(\mathbf{A}^{\alpha\beta}D_\beta\mathbf{u} + \mathbf{b}^\alpha\mathbf{u}\right) + \mathbf{d}^\beta D_\beta\mathbf{u} + \mathbf{V}\mathbf{u},$$

where the principal term,

$$L := -D_\alpha\mathbf{A}^{\alpha\beta}D_\beta$$

is of the type studied by Hofmann and Kim.

Possible tools

The key tools used by Hofmann and Kim:

- Lax-Milgram theorem
- Growth of Dirichlet integrals

Will these tools work when we have lower order terms?

Lax-Milgram theorem

To apply Lax-Milgram, we need a Hilbert space.

The choice of Hilbert space will depend on the properties of the lower order terms.

Furthermore, we need to assume that the associated bilinear form is bounded and coercive in our chosen Hilbert space.

Hilbert space example

Assume that $N = 1$ and

$$\mathscr{L}u := -D_\alpha\left(A^{\alpha\beta}D_\beta u\right) + Vu$$

where, for some $p > \frac{n}{2}$,

$$V \in L^p(\mathbb{R}^n)$$

We choose $H = W^{1,2}(\mathbb{R}^n)$.

Boundedness follows from Hölder inequality and Sobolev embedding.

If $V \geq \varepsilon > 0$ a.e., then we can ensure coercivity.

Further assumptions

The Dirichlet integral assumption that Hofmann and Kim used is not appropriate for non-homogeneous operators.

We use another facet of de Giorgi-Nash-Moser theory, we assume local boundedness of solutions.

Local boundedness

In place of the Dirichlet integral assumption, we assume:

If $\mathbf{u}$ is a weak solution to $\mathscr{L}\mathbf{u} = \mathbf{f}$ or $\mathscr{L}^*\mathbf{u} = \mathbf{f}$ in B_R, for some $R > 0$, where $\mathbf{f} \in L^\ell(B_R)^N$ for some $\ell \in \left(\frac{n}{2}, \infty\right]$, then for any $q > 0$,

$$\sup_{B_{R/2}} |\mathbf{u}| \leq C \left[\left(\frac{1}{|B_R|} \int_{B_R} |\mathbf{u}|^q \right)^{1/q} + R^{2-\frac{n}{\ell}} ||\mathbf{f}||_{L^\ell(B_R)} \right].$$

Note: $\mathscr{L}^*$ denotes the adjoint operator to $\mathscr{L}$.

Scale-invariant local boundedness

In order for our fundamental solution estimates to be on par with those of Hofmann-Kim, we require that C be independent of R.

To accomplish this, we assume positivity of the lower order terms, as well as a Caccioppoli inequality:

If $\mathbf{u}$ is a weak solution to $\mathscr{L}\mathbf{u} = \mathbf{0}$ or $\mathscr{L}^*\mathbf{u} = \mathbf{0}$ in $U \subset \Omega$ and ζ is a smooth cutoff function, then

$$\int |D\mathbf{u}|^2 \zeta^2 \le C \int |\mathbf{u}|^2 |D\zeta|^2,$$

where C is independent of the subdomain U.

Fundamental solutions results

There exists a fundamental matrix, $\mathbf{\Gamma}(x,y) = (\Gamma_{ij}(x,y))_{i,j=1}^N$, on $\{x \neq y\}$, unique in the Lebesgue sense, that satisfies the following estimates:

$$
\begin{aligned}
&||\mathbf{\Gamma}(\cdot,y)||_{L^{\frac{2n}{n-2}}(\mathbb{R}^n \setminus B_r(y))} \leq Cr^{1-\frac{n}{2}} \\
&||D\mathbf{\Gamma}(\cdot,y)||_{L^2(\mathbb{R}^n \setminus B_r(y))} \leq Cr^{1-\frac{n}{2}} \\
&||\mathbf{\Gamma}(\cdot,y)||_{L^q(B_r(y))} \leq C_q r^{2-n+\frac{n}{q}}, \quad \forall q \in \left[1, \tfrac{n}{n-2}\right) \\
&||D\mathbf{\Gamma}(\cdot,y)||_{L^q(B_r(y))} \leq C_q r^{1-n+\frac{n}{q}}, \qquad \forall q \in \left[1, \tfrac{n}{n-1}\right) \\
&|\{x \in \mathbb{R}^n : |\mathbf{\Gamma}(x,y)| > \tau\}| \leq C\tau^{-\frac{n}{n-2}} \\
&|\{x \in \mathbb{R}^n : |D_x\mathbf{\Gamma}(x,y)| > \tau\}| \leq C\tau^{-\frac{n}{n-1}} \\
&|\mathbf{\Gamma}(x,y)| \leq C|x-y|^{2-n}
\end{aligned}
$$

for all $r, \tau > 0$.

Hölder continuity assumption

Assume that whenever $\mathbf{u}$ is a weak solution to $\mathscr{L}\mathbf{u} = \mathbf{0}$ or $\mathscr{L}^*\mathbf{u} = \mathbf{0}$ in B_{R_0} for some $R_0 > 0$, there exists $\eta \in (0,1)$, depending on R_0, and $C_{R_0} > 0$ so that whenever $0 < R \le R_0$,

$$\sup_{x,y \in B_{R/2}, x \neq y} \frac{|\mathbf{u}(x) - \mathbf{u}(y)|}{|x-y|^\eta} \le C_{R_0} R^{-\eta} \left(\frac{1}{|B_R|} \int_{B_R} |\mathbf{u}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}}$$

Consequences of Hölder continuity

If we further assume that all solutions to $\mathscr{L}\mathbf{u} = \mathbf{0}$ are (locally) Hölder continuous, then there exists a unique *continuous* fundamental matrix, $\mathbf{\Gamma}(x, y)$.

Moreover, $\mathbf{\Gamma}(x, y) = \mathbf{\Gamma}^*(y, x)^T$, where $\mathbf{\Gamma}^*$ is the unique continuous fundamental matrix associated to $\mathscr{L}^*$.

More consequences of Hölder continuity

Furthermore, $\mathbf{\Gamma}(x, y)$ satisfies the following estimates:

$$||\mathbf{\Gamma}(\cdot, y)||_{L^{\frac{2n}{n-2}}(\mathbb{R}^n \setminus B_r(y))} + ||\mathbf{\Gamma}(x, \cdot)||_{L^{\frac{2n}{n-2}}(\mathbb{R}^n \setminus B_r(x))} \leq C r^{1-\frac{n}{2}}$$

$$||D\mathbf{\Gamma}(\cdot, y)||_{L^2(\mathbb{R}^n \setminus B_r(y))} + ||D\mathbf{\Gamma}(x, \cdot)||_{L^2(\mathbb{R}^n \setminus B_r(x))} \leq C r^{1-\frac{n}{2}}$$

$$||\mathbf{\Gamma}(\cdot, y)||_{L^q(B_r(y))} + ||\mathbf{\Gamma}(x, \cdot)||_{L^q(B_r(x))} \leq C_q r^{2-n+\frac{n}{q}}, \quad \forall q \in \left[1, \tfrac{n}{n-2}\right)$$

$$||D\mathbf{\Gamma}(\cdot, y)||_{L^q(B_r(y))} + ||D\mathbf{\Gamma}(x, \cdot)||_{L^q(B_r(x))} \leq C_q r^{1-n+\frac{n}{q}}, \qquad \forall q \in \left[1, \tfrac{n}{n-1}\right)$$

$$|\{x \in \mathbb{R}^n : |\mathbf{\Gamma}(x, y)| > \tau\}| + |\{y \in \mathbb{R}^n : |\mathbf{\Gamma}(x, y)| > \tau\}| \leq C \tau^{-\frac{n}{n-2}}$$

$$|\{x \in \mathbb{R}^n : |D_x\mathbf{\Gamma}(x, y)| > \tau\}| + |\{y \in \mathbb{R}^n : |D_y\mathbf{\Gamma}(x, y)| > \tau\}| \leq C \tau^{-\frac{n}{n-1}}$$

$$|\mathbf{\Gamma}(x, y)| \leq C |x - y|^{2-n}, \qquad \forall x \neq y,$$

for all $r, \tau > 0$.

even more...

Moreover, for any $0 < R \leq R_0 < |x - y|$,

$$|\mathbf{\Gamma}(x,y) - \mathbf{\Gamma}(z,y)| \leq C_{R_0} C \left(\frac{|x-z|}{R} \right)^{\eta} R^{2-n}$$

whenever $|x - z| < \frac{R}{2}$ and

$$|\mathbf{\Gamma}(x,y) - \mathbf{\Gamma}(x,z)| \leq C_{R_0} C \left(\frac{|y-z|}{R} \right)^{\eta} R^{2-n}$$

whenever $|y - z| < \frac{R}{2}$.

Green functions

Under a similar set of assumptions, an analogous collection of statements hold for Green functions on *arbitrary* open, connected domains,

$$\Omega \subset \mathbb{R}^n, \qquad \text{where} \quad n \geq 3.$$

Examples

There are a few important examples of non-homogeneous elliptic operators that fit into our framework.

I'll show three examples for the fundamental solution.

Case 1: Homogeneous operators

If $\mathbf{b}, \mathbf{d}, \mathbf{V} \equiv \mathbf{0}$, then

$$\mathscr{L} = L.$$

The Hilbert space for solutions is $\mathbf{H}(\mathbb{R}^n) = Y^{1,2}(\mathbb{R}^n)^N$.

Here, $Y^{1,2}(\Omega)$ is the family of all weakly differentiable functions $u \in L^{2^*}(\Omega)$, with $2^* = \frac{2n}{n-2}$, whose weak derivatives are functions in $L^2(\Omega)$.

Case 2: Lower order coefficients in L^p

Assume

$$\mathbf{V} \in L^p(\Omega)^{N\times N} \quad \text{for some} \quad p \in \left(\frac{n}{2}, \infty\right]$$
$$\mathbf{b} \in L^s(\Omega)^{n\times N\times N} \quad \text{for some} \quad s \in (n, \infty]$$
$$\mathbf{d} \in L^t(\Omega)^{n\times N\times N} \quad \text{for some} \quad t \in (n, \infty]$$

The Hilbert space for solutions is $\mathbf{H}(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)^N$.

The lower-order terms are chosen so that the bilinear form associated to $\mathscr{L}$ is coercive.

Case 3: Reverse Hölder potentials

Assume $\mathbf{V} \in B_p$, the reverse Hölder class, for some $p \in \left[\frac{n}{2}, \infty\right)$.
Take $\mathbf{b}, \mathbf{d} \equiv \mathbf{0}$.

Recall that $V \in B_p$ if V is a.e. non-negative function that satisfies the reverse Hölder inequality:

$$\left(\frac{1}{|B|}\int_B |V|^p\right)^{\frac{1}{p}} \lesssim \frac{1}{|B|}\int_B V.$$

The Hilbert space for solutions is $\mathbf{H}(\mathbb{R}^n) = W_V^{1,2}(\mathbb{R}^n)^N$, a weighted Sobolev space.

Thank you.