Functional models for finite rank perturbations

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at MSRI

January 2017

This talk is based on joint work with S. Treil.
Thanks to Simons Foundation Collaboration Grant for Mathematicians #426258.

Rank one unitary perturbations

- Given unitary U on H, fix unit vector $b \in H$
- All rank one $K = (\,\cdot\,, b)_H \psi =: \psi b^*$ for which perturbed operator $U + K$ is unitary parametrized by complex $|\gamma| = 1$:

$$U_\gamma = U + K = U + (\gamma - 1) b b^* U$$

- WLOG: b *star-cyclic*, i.e. $H = \overline{\text{span}}\{U^k b, (U^*)^k b : k \in \mathbb{Z}_+\}$
- In the spectral representation of U wrt b we have

$$U_\gamma = M_\xi + (\gamma - 1)\mathbf{1}\,\bar{\xi}^* \qquad \text{on } L^2(\mu) \text{ where } \mu = \mu^{U,b}$$

- When $|\gamma| < 1$, then U_γ is a c.n.u. contraction; fix such γ
- Translate the problem into its functional model

Our goal: Carry out this correspondence!

Model theory and the Clark operator Φ

- Unitarily equivalent formulation of rank one contractions U_γ
- A *Clark operator* is a unitary operator $\Phi : \mathcal{K}_\theta \to L^2(\mu)$ so that

$$U_\gamma \Phi = \Phi \mathcal{M}_\theta \quad \text{and} \quad \Phi \mathbf{1} = \alpha b$$

where:

  - There is a 1-1 correspondence $\{\mu, b\} \leftrightarrow \{\theta \in H^\infty(\mathbb{D})\}$
  - The model space $\mathcal{K}_\theta$ is a backward shift invariant subspace of a generally vector-valued (possibly weighted) L^2 space
  - If θ is inner, then $\mathcal{K}_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$ by Beurling
  - $\mathcal{M}_\theta = P_\theta M_z|_{\mathcal{K}_\theta}$ is the compression of the forward shift
  - WLOG $\alpha = 1$

What should Φ^* look like? A formal computation

Take $|\gamma| = 1$ and substitute

$$U_\gamma = U + (\gamma - 1)bb^*U$$

into $M_z\Phi^* = \Phi^*U_\gamma$. Using $\Phi^*b = \mathbf{1}$ we obtain

$$M_z\Phi^* = \Phi^*U_\gamma = \Phi^*[U + (\gamma - 1)bb^*U] = \Phi^*U + (\gamma - 1)\mathbf{1}b^*U.$$

ASSUME that Φ^* is an integral operator with kernel $K(z, \xi)$ and deal only with the integrand:

$$K(z, \xi)z = K(z, \xi)\xi + (\gamma - 1)\xi$$
$$K(z, \xi) = \frac{(\gamma - 1)\xi}{z - \xi} = \frac{1 - \gamma}{1 - \bar{\xi}z}\,.$$

Integrating wrt spectral measure μ, we obtain a Cauchy-type SIO.

Unitary rank one perturbations were studied by Aleksandrov, Ball, Clark, Douglas–Shapiro–Shields, Kapustin, Poltoratski, Ross, Sarasson, etc.

A self-adjoint setting was studied by Albeverio–Kurasov, Aronszajn–Donoghue, delRio, Kato-Rosenblum, Simon, etc.

Finite rank generalizations occur in literature by Albeverio–Kurasov (extension theory), Kapustin–Poltoratski (case $\mu \perp dx$).

Finite rank unitary perturbations

- Consider unitary operator U on separable Hilbert space H
- Assume U is star-cyclic!
- By the spectral theorem, WLOG, assume $U = M_\xi$ in $H = L^2(\mu)$, where μ is a probability measure on $\mathbb{T}$
- We study the family of rank d perturbations $U + K$ with $\operatorname{Ran} K \subset E$ for some a priori fixed $E \le H$, dim $E = d$
- All unitary perturbations $U + K$ of U can be parametrized as

$$T = U + (R - \mathbf{I}_E) P_E U,$$

where R runs over all unitary operators in E

Slight reformulation

The perturbation $T = U + (R - \mathbf{I}_E)P_E U$ can be rewritten as

$$T = U + \mathbf{B}(\Gamma - \mathbf{I}_{\mathbb{C}^d})\mathbf{B}^* U$$

where:

- We factorized R through $\mathbb{C}^d$ by picking an isometric operator $\mathbf{B} : \mathbb{C}^d \to H$, $\operatorname{Ran} \mathbf{B} \subset E$
- A "matrix" representation of $\mathbf{B}$ comes about from defining functions $b_k \in L^2(\mu)$ by $b_k := \mathbf{B} e_k$
- $\mathbf{B}$ is the multiplication by the row vector function

$$B(\xi) = (b_1(\xi), b_2(\xi), \ldots, b_d(\xi))$$

- The adjoint acts by $\mathbf{B}^* f = \int_{\mathbb{T}} f(\xi) B^*(\xi) d\mu(\xi)$
- Then $R = \mathbf{B}\Gamma\mathbf{B}^*$ where Γ is a $d \times d$ matrix

For rank one unitary perturbations this reduces to the usual

$$T = U + (\gamma - 1) b b^* U \quad \text{with } \gamma \in \mathbb{T}.$$

Star-cyclic subspaces and c.n.u. contractions

- Recall: Vector φ is *star-cyclic* for an operator T, if

$$H = \overline{\operatorname{span}}\{T^k\varphi, (T^*)^k\varphi : k \in \mathbb{Z}_+\}$$

- Ex.: Given ONB $\{f_i\}_{i\in\mathbb{N}}$ of H, consider the unitary operator $U = \sum \gamma_i f_i f_i^*$ with $\gamma_i \neq \gamma_j$ for $i \neq j$ and $\gamma_i \in \mathbb{T}$. This operator is star-cyclic with star-cyclic vector $\varphi = \sum 2^{-i} f_i$. Take $E = \operatorname{span}\{f_1, f_2\}$ and consider rank two perturbation $T = \gamma f_1 f_1^* + \gamma f_2 f_2^* + \sum_{i>2} \gamma_i f_i f_i^*$. Operator T has eigenvalue γ with multiplicity 2. T is not star-cylic.
- A subspace E is *star-cyclic* for an operator T on H, if

$$H = \overline{\operatorname{span}}\{T^k E, (T^*)^k E : k \in \mathbb{Z}_+\}$$

Lemma

If $E = \operatorname{Ran}\mathbf{B}$ is a star-cyclic subspace for U and $\|\Gamma\|_2 < 1$, then $T = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^*U$ is a c.n.u. contraction.

Model theory

- For a contraction T its *defect operators* and *defect spaces* are

$$D_T := (\mathbf{I} - T^*T)^{1/2}, \qquad D_{T^*} := (\mathbf{I} - TT^*)^{1/2},$$

$$\mathfrak{D}_T := \operatorname{clos}\operatorname{Ran} D_T, \qquad \mathfrak{D}_{T^*} := \operatorname{clos}\operatorname{Ran} D_{T^*}$$

- *Operator-valued characteristic function* $\theta \in H^\infty(\mathfrak{D}\to\mathfrak{D}_*)$, where $\dim\mathfrak{D} = \dim\mathfrak{D}_T$ and $\dim\mathfrak{D}_* = \dim\mathfrak{D}_{T^*}$
- Here $\dim\mathfrak{D} = \dim\mathfrak{D}_T = d$
- The *model space* $\mathcal{K}_\theta$ is a backward shift invariant subspace of $L^2(\mathfrak{D}_* \oplus \mathfrak{D}; W)$, the *model operator* $\mathcal{M}_\theta$ is the compression of the forward shift
- Sz.-Nagy–Foiaș uses $W \equiv \mathbf{I}$ and with $\Delta := (\mathbf{I}_\mathfrak{D} - \theta^*\theta)^{1/2}$

$$\mathcal{K}_\theta := \begin{pmatrix} H^2(\mathfrak{D}_*) \\ \operatorname{clos}\Delta L^2(\mathfrak{D}) \end{pmatrix} \ominus \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2(\mathfrak{D})$$

- If θ is inner, then $\Delta \equiv \mathbf{0}$. So $\mathcal{K}_\theta = H^2(\mathfrak{D}_*) \ominus \theta H^2(\mathfrak{D})$

Operator-valued characteristic function θ

In the formula for the characteristic function

$$\theta_T(z) = \mathbf{B}^* \left(-T + zD_{T^*} \left(\mathbf{I}_H - zT^*\right)^{-1} D_T\right) (\mathbf{B}^* U)^* \Big|_{\mathfrak{D}}, \quad z \in \mathbb{D}$$

we need to invert a $2d \times 2d$ matrix.

Theorem

Let $T = U + \mathbf{B}(\Gamma - \mathbf{I}_{\mathbb{C}^d})\mathbf{B}^* U$ with strict contraction Γ. In matrix representation the characteristic function $\theta_T \in H^\infty(\mathbb{C}^d \to \mathbb{C}^d)$ is

$$\theta_T(z) = -\Gamma + D_{\Gamma^*} F_1(z)\Big(\mathbf{I}_{\mathbb{C}^d} - (\Gamma^* - \mathbf{I}_{\mathbb{C}^d})F_1(z)\Big)^{-1} D_\Gamma, \text{ where}$$

$$F_1(z) := z\mathbf{B}^*(\mathbf{I}_H - zU^*)^{-1}U^*\mathbf{B} = \int_{\mathbb{T}} \frac{z\overline{\xi}}{1 - z\overline{\xi}} M(\xi)\mathrm{d}\mu(\xi)$$

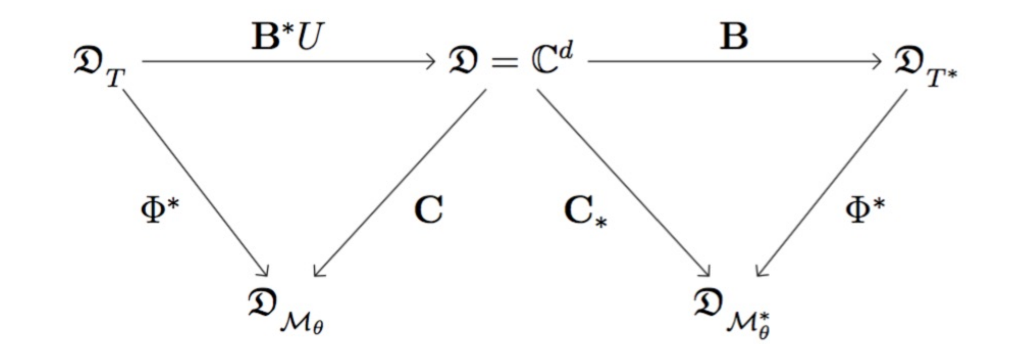
with matrix-valued function M, $M_{j,k}(\xi) = \overline{b_j(\xi)} b_k(\xi)$.

If Γ is also normal, then the characteristic functions θ_Γ and θ are related via linear fractional transformation

$$\theta_\Gamma = D_\Gamma^{-1}(\theta - \Gamma)(\mathbf{I}_{\mathbb{C}^d} - \Gamma^*\theta)^{-1} D_\Gamma.$$

Clark operator

- A *Clark operator* $\Phi : \mathcal{K}_\theta \to H$ is unitary with $\Phi \mathcal{M}_\theta = T\Phi$
- Parametrizations $\mathbf{C} : \mathbb{C}^d \to \mathfrak{D}_{\mathcal{M}_\theta}$ and $\mathbf{C}_* : \mathbb{C}^d \to \mathfrak{D}_{\mathcal{M}_\theta^*}$ *agree*, if there is a Clark operator Φ so that the diagram commutes



Lemma

In the above setting, the following parametrizations agree

$$\mathbf{C}_* e_* = \begin{pmatrix} \mathbf{I} - \theta(z)\theta^*(0) \\ -\Delta(z)\theta^*(0) \end{pmatrix} \left(\mathbf{I} - \theta(0)\theta^*(0)\right)^{-1/2} e_*, \qquad e_* \in \mathbb{C}^d,$$

$$\mathbf{C} e = \begin{pmatrix} z^{-1}\left(\theta(z) - \theta(0)\right) \\ z^{-1}\Delta(z) \end{pmatrix} \left(\mathbf{I} - \theta^*(0)\theta(0)\right)^{-1/2} e, \qquad e \in \mathbb{C}^d.$$

Representation of Φ^*

Operator $\mathbf{C} : \mathbb{C}^d \to \mathfrak{D}_{\mathcal{M}_\theta}$ can be realized as multiplication by $2d \times d$ matrix-valued function: $(\mathbf{C}e)(z) = C(z)e$ for $e \in \mathbb{C}^d$. And let $C_*(z)$ be so that $(\mathbf{C}_* e_*)(z) = C_*(z)e_*$ for $e_* \in \mathbb{C}^d$.

Theorem (Universal representation)

Consider the usual finite rank perturbation setting:

- $T = U + \mathbf{B}(\Gamma - \mathbf{I}_{\mathbb{C}^d})\mathbf{B}^* U$ *with* $\|\Gamma\| < 1$ *and* $U = M_\xi$ *in* $H = L^2(\mu)$
- $\operatorname{Ran} B$ *star-cyclic;* θ *characteristic function of* T*;* $\mathcal{K}_\theta$ *model space*
- $\mathbf{C}$ *and* $\mathbf{C}_*$ *parameterizing unitary operators agree* $\Phi : \mathcal{K}_\theta \to H$.
- *And let* $C(z)$, $C_*(z)$ *be as above.*

Then for all $b \in \operatorname{Ran} \mathbf{B}$ *and for all* $f \in C^1(\mathbb{T})$ *we have*

$$\big(\Phi^* f b\big)(z) = f(z)C_*(z)\mathbf{B}^* b + C_1(z) \int \frac{f(\xi) - f(z)}{1 - z\bar{\xi}} B^*(\xi) b(\xi) d\mu(\xi)$$

with $C_1(z) = C_*(z) - zC(z)$, $B^*(\xi) = \left(\overline{b_1(\xi)}, \overline{b_2(\xi)}, \dots, \overline{b_d(\xi)}\right)^\top$ *and* $\mathbf{B}^* b = \int B^*(\xi) b(\xi) d\mu(\xi)$.

Regularization

Fix $f \in L^2(\mu)$ and $g \in L^2(\mathbb{C}^d \oplus \mathbb{C}^d)$ so that f and g have compact supports, $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$. Through approximation

$$(\Phi^* f, g) = \int_{\mathbb{T}} g^*(z) C_1(z) \int_{\mathbb{T}} \frac{f(\xi)}{1 - z\bar{\xi}} B^*(\xi) d\mu(\xi) dm(z).$$

Those terms of the representation formula for Φ^* that include $f(z)$ vanish because of the separated support assumptions.

$$|(\Phi^* f, g)| \leq \|f\|_{L^2(\mu)} \|g\|_{L^2(\mathbb{C}^{2d})}.$$

Theorem

Let μ and ν be Radon measures in $\mathbb{R}^N$ without common atoms. Assume that a kernel $K \in L^2_{\mathrm{loc}}(\mu \times \nu)$ is L^p restrictedly bounded, with the restricted norm C. Then the integral operator with T kernel K is a bounded operator $L^p(\mu) \to L^p(\nu)$ with the norm at most $2C$.

Through standard mollification $(T_r^{B^*\mu} g)(z) = \int_{\mathbb{T}} \frac{g(\xi)}{1 - rz\bar{\xi}} B^*(\xi) d\mu(\xi)$ is bounded with norm independent of r.

Φ^* on all of $L^2(\mu)$

- The Cauchy-type operator $(T_r^{B^*\mu} g)(z) = \int_{\mathbb{T}} \frac{g(\xi)}{1 - rz\bar{\xi}} B^*(\xi) d\mu(\xi)$ has L^2 boundary values as $r \to 1-$ for a.e. $z \in \mathbb{T}$
- Define pointwise $(T_\pm^{B^*\mu} g)(z) := \lim_{r \to 1\mp} (T_r^{B^*\mu} g)(z)$
- Operators $C_1 T_\pm^{B^*\mu} : L^2(\mu) \to L^2(\mathbb{C}^d \oplus \mathbb{C}^d)$ are bounded and

$$C_1 T_\pm^{B^*\mu} = \text{w.o.t.-} \lim_{r \to 1\mp} C_1 T_r^{B^*\mu}$$

Theorem

In Sz.-Nagy–Foiaș transcription Φ^* is represented for $f \in L^2(\mu)$ by

$$(\Phi^* f)(z) = C_1(z)(T_+^{B^*\mu} f)(z) + f(z)\Psi(z), \quad z \in \mathbb{T}, \text{ where}$$

$$\Psi(z) := b^{-1}(z)[C_*(z)\mathbf{B}^* b - C_1(z)(T_+^{B^*\mu} b)(z)].$$

Summary

- Defined finite rank perturbations and star-cyclic subspaces
- Introduced model spaces and Clark operator
- Matrix-valued characteristic functions related by linear fractional transformations, if Γ is normal
- Universal representation for Φ_γ^* on $C^1(\mathbb{T})$ and ...
- ... on all of $L^2(\mu)$ via regularization in Sz.-Nagy–Foiaș transcription

Some possible future questions

- Allow non-simple U
- Investigate cyclicity properties
- Perturbation theory (expressing μ_Γ in terms of μ)
- Infinite rank perturbations T with trace class D_T
- Ramifications for Anderson-type Hamiltonians (infinite rank random perturbations includes the discrete random Schrödinger operator)