# Radial Fourier Multipliers

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# Fourier Multiplier Operators

- Fourier multiplier operators are a basic object of study in harmonic analysis.
- Given $m \in L^\infty(\mathbb{R}^d)$, we may define an operator $T_m$ acting on Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{F}[T_m f](\xi) = m(\xi)\widehat{f}(\xi).$$

- One is typically interested in the mapping properties of $T_m$ between various function spaces.
- Most basic question to ask: For a given $p$, does $T_m$ extend to a bounded operator on $L^p$?

# $L^p$ Mapping Properties of Multipliers

- Is there a *characterization*, i.e. a simple and useful criterion for $m$ that determines $L^p$ boundedness of $T_m$ for general multipliers $m \in L^\infty$?
- If $p = 1$, $T_m$ is bounded on $L^1$ if and only if $m$ is the Fourier transform of a finite Borel measure.
- If $p = 2$, $T_m$ is bounded on $L^2$ by Plancherel since $m \in L^\infty$.
- If $p \neq 1, 2$, it is widely believed that no reasonable characterization exists.
- What if we ask the same question but restrict the class of multipliers $m \in L^\infty$ to a smaller subclass, for example the subclass of bounded, *radial* functions?

## The Radial Fourier Multiplier Conjecture (Simplified Version)

*Let* $1 < p < p_d := \frac{2d}{d+1}$ *and* $d \geq 2$. *If* $m \in L^\infty(\mathbb{R}^d)$ *is radial and supported in a compact subset of* $\{\xi : 1/2 < |\xi| < 2\}$, *then the operator* $T_m$ *is bounded on* $L^p(\mathbb{R}^d)$ *if and only if* $K := \widehat{m} \in L^p(\mathbb{R}^d)$. *Moreover, we actually have*

$$\|T_m\|_{L^p \to L^p} \approx_p \|K\|_p \,. \tag{1}$$

- If (1) is true, we will say $\mathrm{Rad}(d, p)$ holds.

## The Radial Fourier Multiplier Conjecture (Full Version)

*Let* $1 < p < p_d := \frac{2d}{d+1}$ *and* $d \geq 2$. *Fix an arbitrary Schwartz function* $\eta$ *that is not identically* $0$. *If* $m \in L^\infty(\mathbb{R}^d)$ *is radial, then*

$$\|T_m\|_{L^p \to L^p} \approx_p \sup_{t>0} t^{d/p} \|T_m[\eta(t\cdot)]\|_{L^p} \,.$$

# Background and Motivation for the Conjecture

In 2008, Garrigós and Seeger obtained a characterization of $L^p_{\text{rad}}$ boundedness of radial Fourier multipliers, where $L^p_{\text{rad}}$ denotes the space of radial $L^p$ functions. The simplified version of their result states:

**Theorem (Garrigós and Seeger, 2008)**

*Let $1 < p < p_d := \frac{2d}{d+1}$ and $d \geq 2$. If $m \in L^\infty(\mathbb{R}^d)$ is radial and supported in a compact subset of $\{\xi : 1/2 < |\xi| < 2\}$, then the operator $T_m$ is bounded on $L^p_{rad}(\mathbb{R}^d)$ if and only if $K := \widehat{m} \in L^p(\mathbb{R}^d)$. Moreover, we actually have*

$$\|T_m\|_{L^p_{rad} \to L^p_{rad}} \approx_p \|K\|_p .$$

# Previous partial progress toward the Conjecture for $d \geq 4$

In 2011, Heo, Nazarov, and Seeger proved the conjecture in the partial range $1 < p < \frac{2d-2}{d+1}$ in dimensions $d \geq 4$. They actually proved the following stronger conjecture in the partial range $1 < p < \frac{2d-2}{d+1}$ and $d \geq 4$.

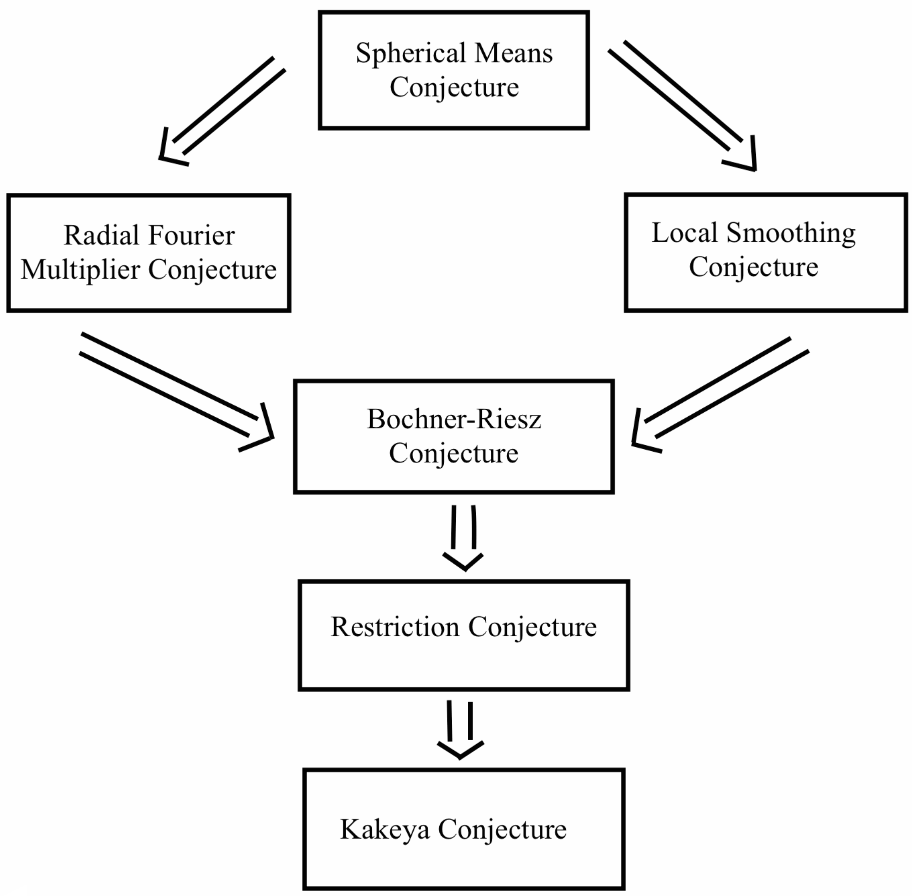
**The Spherical Means Conjecture**

*Let $1 < p < p_d := \frac{2d}{d+1}$. Let $\sigma_r$ denote the surface measure on the $(d-1)$-sphere of radius $r$ centered at the origin. Let $\psi_0$ be a smooth, radial function supported in the unit ball centered at the origin whose Fourier transform vanishes to higher order (say $100d$) at the origin. Set $\psi = \psi_0 * \psi_0$. There is a constant $C_p$ so that for every $h \in L^p(\mathbb{R}^d \times \mathbb{R}^+; dy\, r^{d-1}\, dr)$ we have*

$$\left\| \int_{\mathbb{R}^d} \int_1^\infty h(y,r)\, \sigma_r * \psi(\cdot - y)\, dr\, dy \right\|_{L^p(\mathbb{R}^d)} \leq C_p \left( \int \int_{\mathbb{R}^d \times \mathbb{R}^+} |h(y,r)|^p\, dy\, r^{d-1}\, dr \right)^{1/p}. \quad (2)$$

- If (2) is true we will say $\mathrm{Sph}(d,p)$ holds.

# A Tree of Conjectures in Harmonic Analysis



# New Results in Three and Four Dimensions

We improve Heo, Nazarov, and Seeger's range for Rad$(4, p)$ from $1 < p < 6/5$ to $1 < p < 36/29$.

**Theorem 1 (C., 2016)**

*The Radial Fourier Multiplier Conjecture in four dimensions holds in the range $1 < p < 36/29$.*

In three dimensions, we obtain a characterization in the range $1 < p < 13/12$ in terms of the $L^{p,1}$ norm of the kernel.

**Theorem 2 (C., 2016)**

*Let $1 < p < 13/12$. Let $m \in L^\infty(\mathbb{R}^3)$ be radial and supported in a compact subset of $\{\xi : 1/2 < |\xi| < 2\}$. Then $T_m$ is restricted strong type $(p, p)$ if $K = \widehat{m} \in L^p(\mathbb{R}^3)$, and $T_m$ is bounded on $L^p(\mathbb{R}^3)$ if $K \in L^{p,1}(\mathbb{R}^3)$. Moreover*

$$\|T_m\|_{L^p \to L^p} \lesssim \|K\|_{L^{p,1}} .$$

- We expect $\|K\|_{L^{p,1}}$ in the second theorem could be improved to $\|K\|_{L^p}$, which would imply that the Radial Fourier Multiplier Conjecture holds in the range $1 < p < 13/12$ in $\mathbb{R}^3$.

## Preliminaries to the proof of Theorem 2

- First, some motivation for what we are about to do next:
- Since the multiplier $m$ is compactly supported away from the origin, we have $\widehat{m} =: K = K * \phi$ where $\phi$ is a smooth bump with $\hat{\phi}$ supported in a compact set away from the origin (which implies $\phi$ has a lot of cancellation).
- Moreover, since $\widehat{K}$ is supported in the double of the unit ball, $K$ is "essentially constant" at unit scales, and so since it is radial it is essentially constant on annuli centered at the origin of thickness 1.
- Thus we should expect that we should be able to "decompose" the kernel K into functions that have a lot of cancellation and are supported on annuli of thickness $\approx 1$ centered at the origin.

# Preliminaries to the proof of Theorem 2

- Some machinery of Heo, Nazarov, and Seeger is needed.
- As in [HNS], the first step is to **discretize** the problem and to reduce it to proving an inequality involving sums of functions with cancellation supported on annuli of thickness $\approx 1$ whose centers and radii lie in a discrete set.
- Let $\mathcal{Y} \subset \mathbb{R}^3$ be the integer lattice in $\mathbb{R}^3$ which will represent **centers** of 3-D annuli and let $\mathcal{R} \subset \mathbb{R}$ be the integers, which will represent **radii** of 3-D annuli.
- For $(y, r) \in \mathcal{Y} \times \mathcal{R}$, let $F_{y,r}$ denote the function $\psi * \sigma_r(\cdot - y)$, where $\psi$ is a smooth compactly supported function whose Fourier transform vanishes to high order at the origin and where $\sigma_r$ is the surface measure on the 2-sphere of radius $r$ centered at the origin.
- Discretization, followed by an application of a dyadic interpolation lemma, reduces Sph$(3, p)$ to proving the following inequality for every finite set $\mathcal{E} \subset \mathcal{Y} \times \mathcal{R}$ and every measurable function $c : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$ with $|c(y, r)| \leq 1$:

$$\left\| \sum_{(y,r)\in\mathcal{E}} c(y, r) F_{y,r} \right\|_p^p \lesssim_p \sum_k 2^{2k} \# \mathcal{E}_k, \tag{3}$$

where $\mathcal{E}_k = \mathcal{E} \cap (\mathcal{Y} \times [2^k, 2^{k+1}])$.

# Preliminaries to the proof of Theorem 2

Density decompositions

- For dyadic numbers $u \geq 1$, we decompose $\mathcal{E}_k$ into sets $\mathcal{E}_k(u)$ of "density" $u$ as follows.
- Set $\widehat{\mathcal{E}_k}(u) := \{(y,r) \in \mathcal{E}_k : \exists$ a ball $B$ of radius $\leq 2^k$ containing $(y,r)$ such that $\#(\mathcal{E}_k \cap B) \geq u(\mathrm{rad}(B))\}$.
- Set $\mathcal{E}_k(u) = \widehat{\mathcal{E}_k}(u) \setminus \bigcup_{u'>u \text{ dyadic}} \widehat{\mathcal{E}_{k'}}(u)$.
- We have

$$\mathcal{E}_k = \bigcup_{u \geq 1 \text{ dyadic}} \mathcal{E}_k(u).$$

- For a given function $c(y,r) : \mathcal{Y} \times \mathcal{R} \to \mathbb{C}$, set

$$G_{u,k} = \sum_{(y,r) \in \mathcal{E}_k(u)} c(y,r) F_{y,r},$$

$$G_u = \sum_k G_{u,k}.$$

# Preliminaries to the proof of Theorem 2

## $L^2$ bounds vs. support size

**Lemma (Support size estimate, [HNS])**

*For all dyadic* $u \geq 1$*, the Lebesgue measure of the support of* $G_{u,k}$ *is* $\lesssim u^{-1} 2^{2k} \# \mathcal{E}_k$.

- To prove the restricted strong type version of $\mathrm{Rad}(3, p)$, it actually suffices to prove inequality (3) with the additional assumption that $\mathcal{E}$ is a **product**, i.e. a set of the form $Y \times R$ where $Y \subset \mathcal{Y}$ and $R \subset \mathcal{R}$. Under this assumption, we obtain the following $L^2$ estimate, which is an improvement over the $L^2$ estimate proved in [HNS].

**Lemma (Improved $L^2$ bound)**

*For all* $\epsilon > 0$,

$$\| G_u \|_2^2 \lesssim_\epsilon u^{11/13+\epsilon} \sum_k 2^{2k} \# \mathcal{E}_k.$$

(Recall $G_u := \sum_k \sum_{(y,r) \in \mathcal{E}_k(u)} c(y,r) F_{y,r}$.)

## Outline of proof of the $L^2$ estimate

- We require **scalar product estimates**:

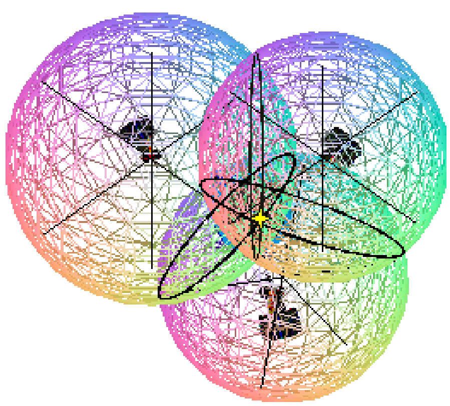
$$|\langle F_{y,r}, F_{y',r'}\rangle| \lesssim \frac{r\,r'}{(1+|y-y'|+|r-r'|)} \times \sum_{\pm,\pm}(1+r\pm r'\pm|y-y'|)^{-N}$$

  for any $N>0$.
- For a fixed $y, r, r'$ the set of $y'$ for which the second term in the product is the worst is contained in the union of two annuli centered at $y$, one of radius $r+r'$ and one of radius $|r-r'|$.
- This corresponds exactly to **tangencies** of annuli.
- We will use a geometric argument to control the number of tangencies between annuli.
- For our argument, it will be essential to use the product structure of the set $\mathcal{E}$.

# A geometric lemma

**Lemma**

*Fix integers $m, l$ with $l \leq m$. Fix $t \approx 2^m$. Then the size of the intersection of three annuli in $\mathbb{R}^3$ of thickness $\approx 1$ and inner radius $t$ such that the distance between the centers of any pair is at least $2^l$ and no greater than $2^m/10$ is $\lesssim 2^{3(m-l)}$, provided that $l \geq m/2 + 10$.*



Thank you!