Sharp(ened) Affine-Invariant Inequalities

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Talk I

short course

affine invariance inequalities

$|E| = \#$ pts. in E

$T: L^p \to L^q$

possible questions:
- op. norm?
- norm attained by some functions?
- which functions maximize $\frac{\|Tf\|_q}{\|f\|_p}$
- stability:
- ⋮

Let's consider inequalities invariant under affine automorphisms

Important inequalities:
- Brunn-Minkowski (Sums of Sets)
- Young's convolution (multiplicity)
- Symmetrization - Riesz-Sobolev
- F. transform Haus. Young

upper bound

$1_A * 1_B = \#$ ways to represent x as $A+B$

$|A+B| \leq |A| \cdot |B|$

lower

$|A+B|^{1/d} \geq |A|^{1/d} + |B|^{1/d}$ for $A, B \neq \emptyset$ in $\mathbb{R}^d$

(Brunn 1887, Lusternik 1935)

notice this says $|A+A| \geq 2^d |A|$
"goes up w/ dimension"

$$\|f * g\|_r \leq \|f\|_p \|g\|_q \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

(Young 1913)

an improved constant exists (Beckner, Brascamp-Lieb 1974)

$$G(x) := c \cdot e^{-Q(x,y)} e^{x \cdot y}$$

"Gaussian"

Affine transf. map convex → convex
ellipsoids → ellipsoids

Q What other classes of sets are preserved?

A Arithm. progressions

$A, B, C \subseteq \mathbb{R}^d$ measurable

$$\langle 1_A * 1_B, 1_C \rangle \leq \langle 1_{A^*} * 1_{B^*}, 1_{C^*} \rangle$$

$|A| = |A^*|$ — closed ball at origin

(Riesz-Sobolev 1930 1938)

$f \geq 0 \qquad f = \int_0^\infty 1_{E_t}\, dt$ "Layer-Cake"

(E_t: indicators $\{f > t\}$.)

When is this an equality? ...

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equality for Young's & Brunn-Minkowski

$$\|\hat{f}\|_q \leq \|f\|_p \qquad 1 \leq p \leq 2, \quad q = p'$$

Hausdorff did not interpolate.

"It is worth to look at his proof" - Christ

Thus far, the stated results have depended on additive combinatorics.

M.C. ③

- (Let $K < \infty$, $|A|^{\#\emptyset} < \infty$, $\#(A+A) \le K \cdot \#(A)$) $\longrightarrow$ Freiman (A contained in a multi-progression s.t. rank $\le C_K$, Cardinality $\le C_K \#(A)$)

Challenge: Prove an anolog for Leb. m set.

- ($A, B \subseteq \mathbb{R}$, w/ (+) Lebesgue measure. If $|A+B| \le |A| + |B| + \delta$) $\longrightarrow$

Riesz - Sobolev maximized for balls.

Sharpened version: $\displaystyle\int_{E^3} 1_{E_1} * 1_{E_2} \le \int_{(E^3)^*} 1_{E_1^*} * 1_{E_2^*} - c$

Burchard's uniqueness is a corollary

Which sets have the largest Fourier transform?

E maximizes $\dfrac{\| \widehat{1}_E \|_q}{|E|^{1/p}} \Longleftrightarrow E$ ellipsoid.

Sharp(ened) Affine-invariant Inequalities

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MSRI short course
January 23 and 25, 2017

- Today's talk will be an overview of several interrelated inequalities and associated ideas.
- It will be followed on Wednesday by a discussion of the proof of one theorem.

Notations: L^p norms, Lebesgue measure

- $|E|$ will denote Lebesgue measure of set E.
- Regard $|E|$ as counting number of points in E.
- $\mathbf{1_E}$ denotes the indicator function of a set;

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Extremal problems

Let $T : L^p \to L^q$ be a bounded linear operator. One can ask:

- What is the operator norm of T?
- Is this norm attained by some function?
- Which functions maximize $\frac{\|Tf\|_q}{\|f\|_p}$?
  Are maximizers unique, up to natural symmetries?
- (*Stability*) If $\|Tf\|_q \geq (1-\delta)\|f\|_p$ for small δ,
  must f be close in norm to some exact maximizer?
- (*Quasi-extremizers*) If $\|Tf\|_q \geq \delta\|f\|_p$,
  must f have a piece with some definite structure?

- One cannot hope to answer these questions for general operators.
- This microcourse is concerned with certain fundamental inequalities with a great deal of structure.
- Our inequalities are invariant under the full group of **affine** automorphisms of $\mathbb{R}^d$ (or some slight variant).
(The affine group is generated by translations together with invertible linear transformations.)
- Such a high degree of symmetry is rare.

Comment on the role of the affine group

- One parameter subgroups are fundamental building blocks of Lie group structures.

- The affine group acts on the set of all cosets of its one parameter subgroups of $\mathbb{R}^d$.

Some topics and associated inequalities

1. Sums of sets (Brunn-Minkowski)
2. Convolution (Young); multiplicities for sumsets
3. Symmetrization (Riesz-Sobolev)
4. Fourier transform (Hausdorff-Young)

Sumsets

- The sum $A + B$ of two nonempty subsets A, B of an Abelian group is

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

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- **Convolution** of two indicator functions is:

$$\begin{aligned}(\mathbf{1}_A * \mathbf{1}_B)(x) &= \int_{a+b=x} \mathbf{1}_A(a)\mathbf{1}_B(b)\, d\mu \\ &= \text{“number” of ways of representing } x \text{ as } a + b\end{aligned}$$

- $\mathbf{1}_A * \mathbf{1}_B$ is thus a weighted sumset.

On the size of a sumset

- If A, B are finite sets with M, N points then $A + B$ can have as many as MN points.
- There is **no upper bound** for Lebesgue measure $|A + B|$ in terms of $|A|, |B|$ alone; the sum of two null sets can have positive measure.
- Example: The sum of two non-parallel lines in $\mathbb{R}^2$ equals $\mathbb{R}^2$.

Brunn-Minkowski inequality

- **Additive** **lower** bound for continuum sumsets:

$$|\mathbf{A}+\mathbf{B}|^{1/d} \boxed{\geq} \ |A|^{1/d}+|B|^{1/d}$$

  for nonempty $A, B \subset \mathbb{R}^d$.
  [Brunn 1887] for convex sets
  [Lusternik 1935] for general sets

- The sum of two balls is a ball; their radii add.

- The 1D inequality takes a universal form, $|A+B| \geq |A|+|B|$, while the $\mathbb{R}^d$ inequality involves a parameter.

- $|A+A| \geq 2^d|A|$; the lower bound increases with the dimension.

Equality in the Brunn-Minkowski inequality

- $|A + B|^{1/d} = |A|^{1/d} + |B|^{1/d}$ iff A, B are **a pair of homothetic convex sets**

- [Minkowski 1910] for convex sets
  Stated for general sets by [Lusternik 1935]
  Proved by [Henstock-Macbeath 1953]

- "Homothetic" means: related by translations and isotropic dilations.

Young's convolution inequality for LCA groups

- [Young 1913]:

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

whenever $p, q, r \in [1, \infty]$ and $r^{-1} = p^{-1} + q^{-1} - 1$.

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whenever $p, q, r \in [1, \infty]$ and $r^{-1} = p^{-1} + q^{-1} - 1$.

- This can also be viewed as a symmetric trilinear inequality

$$\int \prod_{j=1}^{3} f_j(x_j)\, d\lambda(\mathbf{x}) \leq \prod_{j=1}^{3} \|f_j\|_{p_j} \quad \text{if } \textstyle\sum_j p_j^{-1} = 2.$$

(λ is natural measure on $\{\mathbf{x} : x_1 + x_2 + x_3 = 0\}$.)

Young's convolution inequality for $\mathbb{R}^d$

- For functions on $\mathbb{R}^d$

$$\|f * g\|_r \leq \mathbf{A}^d \|f\|_p \|g\|_q$$

where $\mathbf{A} < 1$ when $p, q, r \in (1, \infty)$

- $$\mathbf{A} = C_p C_q C_r^{-1}$$

where $C_s = s^{1/2s}/t^{1/2t}$ with $t = s'$.
[Beckner and Brascamp-Lieb 1974/75]

- Maximizers are (compatible) ordered pairs of Gaussians.
[Brascamp-Lieb]

Gaussians

In this course, a "Gaussian" is

$$G(x) = ce^{-Q(x,x)}e^{x \cdot v}$$

where Q is a positive definite *real* symmetric quadratic form and $v \in \mathbb{C}^d$.

Young's inequality (continued)

• Beckner's original analysis developed a connection with the Central Limit Theorem.

• There exists a (nonlinear heat) *flow* $[0, \infty) \ni t \mapsto (f_t, g_t, h_t)$ under which $\langle f_t * g_t, h_t \rangle$ is nondecreasing, $f_0 = f$ etc., f_t tends in a suitable sense to a scaled Gaussian as $t \to \infty$. The inequality follows by evaluating the optimal constant for the limiting Gaussians.

• The heat flow analysis was developed by Carlen-Lieb-Loss for a subclass of Hölder-Brascamp-Lieb multilinear inequalities, and by Bennett-Carbery-Tao circa 2008 for general HBL inequalities.

Affine symmetry

Question. Affine transformations map convex sets to convex sets, and ellipsoids to ellipsoids. Are there other natural classes of sets that are preserved, and hence might be expected to intervene in the analysis of inequalities with affine symmetry?

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Question. Affine transformations map convex sets to convex sets, and ellipsoids to ellipsoids. Are there other natural classes of sets that are preserved, and hence might be expected to intervene in the analysis of inequalities with affine symmetry?

Answer. Yes: Arithmetic progressions (and multiprogressions). These are at the heart of proofs of some of the results that I will state in today's lecture. However, other aspects of the subject will instead be emphasized in this (micro)course.

Symmetrization

• For $E \subset \mathbb{R}^d$, $E^\star$ is the (closed) ball centered at 0 with $|E^\star| = |E|$.

• One may be tempted to view $E^\star$ as the most symmetric of all sets with measures equal to $|E|$, having large symmetry group $O(d)$. But any **ellipsoid** is invariant under a subgroup of the affine group that is conjugate to $O(d)$, so is just as symmetric.

Riesz-Sobolev inequality (1930, 1938)

For any Lebesgue measurable sets $A, B, C \subset \mathbb{R}^d$ with finite Lebesgue measures,

$$\langle \mathbf{1_A} * \mathbf{1_B}, \mathbf{1_C} \rangle \leq \langle \mathbf{1_{A^\star}} * \mathbf{1_{B^\star}}, \mathbf{1_{C^\star}} \rangle.$$

This can be read in two ways:

• Upper bound in terms of measures of A, B, C

• Among sets with prescribed measures, maximum is attained by balls centered at 0.

Layer cake principle

Any nonnegative function can be expressed as a superposition of indicator functions:

$$f = \int_0^\infty \mathbf{1}_{E_t}\, dt$$

where E_t are the superlevel sets

$$E_t = \{x : f(x) > t\}.$$

Symmetries of the Riesz-Sobolev inequality

If

$$A_j = \psi(E_j) + v_j$$

where

- $\mathbf{v} \in (\mathbb{R}^d)^3$ satisfies $v_3 = v_1 + v_2$,
- $\psi : \mathbb{R}^d \to \mathbb{R}^d$ is measure-preserving and linear,

then

$$\langle \mathbf{1}_{A_1} * \mathbf{1}_{A_2}, \mathbf{1}_{A_3} \rangle = \langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{E_3} \rangle.$$

Equality in Riesz-Sobolev

For dimension $\boxed{d = 1}$: An ordered triple of sets $\mathbf{E} = (E_1, E_2, E_3)$ satisfies

$$\langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{E_3} \rangle = \langle \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2^\star}, \mathbf{1}_{E_3^\star} \rangle$$

if and only if (up to null sets)

$\boxed{E_n \text{ are } \textbf{intervals}}$ whose centers satisfy $\boxed{c_3 = c_1 + c_2}$

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if and only if (up to null sets)

$\boxed{E_n \text{ are } \textbf{intervals}}$ whose centers satisfy $\boxed{c_3 = c_1 + c_2}$

provided

$$|E_k| \leq |E_i| + |E_j| \qquad \boxed{\textbf{called admissibility}}$$

for all permutations (i, j, k) of $(1, 2, 3)$. [Burchard 1998].

Admissibility is needed for a meaningful characterization

- If E_3 properly contains $E_1 + E_2$ then equality holds, yet nothing can be said about $E_3 \setminus (E_1 + E_2)$.

- $|E_1 + E_2|^{1/d}$ can be as small as $|E_1|^{1/d} + |E_2|^{1/d}$.

Equality in Riesz-Sobolev (Burchard's Thm – continued)

For $\boxed{d > 1}$,

$$\langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{E_3} \rangle = \langle \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2^\star}, \mathbf{1}_{E_3^\star} \rangle$$

occurs

- Only for (homothetic, compatibly centered) **ellipsoids** in the strictly admissible case $|E_k|^{1/d} < |E_i|^{1/d} + |E_j|^{1/d}$,
- Only for (homothetic, compatibly centered) **convex sets** in the borderline admissible case.

Near-maximizers

- **Brunn-Minkowski inequality**: If

$$|\mathbf{A}+\mathbf{B}|^{\mathbf{1/d}} \leq (\mathbf{1}+\delta)\big(|\mathbf{A}|^{\mathbf{1/d}}+|\mathbf{B}|^{\mathbf{1/d}}\big)$$

for small δ then A, B are contained in homothetic convex sets whose measures are larger by factors $1 + o_\delta(1)$.
[Figalli-Jerison 2011] and [C 2011]

- **Young's convolution inequality**: If

$$\|f * g\|_r \geq (1-\delta)\mathbf{A}_{pq}^d \|f\|_p \|g\|_q \qquad \text{for small } \delta > 0$$

then there exists a Gaussian F satisfying $\|f - F\|_p \leq o_\delta(1)\|f\|_p$.
[C 2011]

Hausdorff-Young inequality

- $\boxed{\|\widehat{f}\|_{L^q} \leq \|f\|_{L^p}}$ whenever $1 \leq p \leq 2$, $q = p' = \frac{p}{p-1} =$ conjugate exponent.
- W. H. Young [1913] for $q = 4, 6, 8, \ldots$;
  Hausdorff [1923] for general exponents.

Historical note: Hausdorff did not interpolate.

Hausdorff-Young sharp constant and maximizers

- **For** $\mathbb{R}^d$, $\|\widehat{f}\|_{L^q} \leq \mathbf{A}_p^d \|f\|_{L^p}$ with optimal constant

$$\mathbf{A_p} = \mathbf{p^{1/2p}\, q^{-1/2q}} < 1.$$

- Babenko [1961] for $q = 4, 6, 8, 10, \ldots$;
  Beckner [1975] for all $p \in (1, 2)$.
- All **Gaussian** functions are maximizers;

$$\mathbf{G(x) = ce^{-Q(x)+x\cdot v}}$$

where Q is a positive definite homogeneous *real*–valued quadratic polynomial, and $v \in \mathbb{C}^d$.

Uniqueness

- Lieb [1990] showed that all extremizers are Gaussians.
- His proof exploited symmetry considerations, together with functorial properties of Gaussians.

Compactness and Stability

[C 2014] used ideas adapted from additive combinatorics to prove a compactness theorem:

Theorem. If $\|f_\nu\|_p = 1$ and $\|\widehat{f_\nu}\|_q \to \mathbf{A}_p^d$ then there exists a subsequence (ν_k) and an associated sequence of Lebesgue-measure-preserving affine automorphisms ψ_k of $\mathbb{R}^d$ such that the sequence $(f_{\nu_k} \circ \psi_k : k \in \mathbb{N})$ converges in L^p norm.

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Corollary. If $\|\widehat{f}\|_q \geq (1-\delta)\mathbf{A}_p^d \|f\|_p$ then there exists a Gaussian F satisfying

$$\|f - F\|_p \leq o_\delta(1)\|f\|_p.$$

This $o_\delta(1)$ is made more precise below.

Proofs of all of the results on near-maximizers stated thus far have rested in part on input from additive combinatorics — descriptions of finite sets with relatively small sumsets.

A Pillar of Additive Combinatorics: Freĭman's Theorem

Theorem: [195x] Let $K < \infty$. If a finite nonempty set A has

$$\#(\mathbf{A} + \mathbf{A}) \leq \mathbf{K} \cdot \#(\mathbf{A}),$$

then A is contained in a multi-progression satisfying

$$\begin{cases} \textbf{rank} & \leq C_K \\ \textbf{cardinality} & \leq C_K \, \#(A). \end{cases}$$

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This holds for arbitrarily large constants K.

- Central in Gowers' work on Szemerédi's theorem.
- Key to (one) proof: Fourier analysis (Parseval plus theory of Bohr sets) is used to construct an associated multiprogression.
- Corollary via Lebesgue's differentiation theorem: Corresponding result for subsets of $\mathbb{R}^d$, with Lebesgue measure replacing cardinality.

Continuum version of Freĭman's [195x] little theorem

Theorem.
Let $A, B \subset \mathbb{R}^1$ be Borel sets with finite Lebesgue measures. If

$$|\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}| + \delta$$

and

$$\delta < \min(|A|, |B|)$$

then A is contained in an **interval** of length $\leq |A| + \delta$.

Sharpened Hausdorff-Young inequality

Theorem. If $\|f\|_p = 1$ then

$$\|\widehat{f}\|_q \;\le\; \mathbf{A}_p^d \,\|f\|_p - \mathbf{c_{d,p}}\ \text{distance}(\mathbf{f}, \mathfrak{G})^{\mathbf{2}}.$$

- $\mathfrak{G}$ is the set of all Gaussians.
- $c_{d,p} > 0$.
- Distance is measured in L^p norm; distance$(f, \mathfrak{G}) = \min_{G\in\mathfrak{G}} \|f - G\|_{L^p}$.
- The exponent 2 is optimal.
- The functional $f \mapsto \|\widehat{f}\|_q \,/\, \|f\|_p$ is *not* twice continuously differentiable on L^p.

Fourier transform and convolution are related

- $$\|\widehat{f}\|_{2m}^{2m} = \|f * f * \cdots * f\|_2^2$$

  with m factors in the convolution product.

- This connection with convolution, together with the layer cake decomposition, brings sumsets into the discussion. Freĭman's (first!) theorem can thus be exploited. That is the starting point of the proof.

- The upper bound C_K on the rank of the multiprogression in the conclusion of that theorem is one key step towards precompactness of extremizing sequences.

We now turn to the Riesz-Sobolev inequality

$$\langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{E_3} \rangle \leq \langle \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2^\star}, \mathbf{1}_{E_3^\star} \rangle.$$

Sharpened Riesz-Sobolev inequality

Theorem. For any $d \geq 1$ and any $\mathbf{E} = (E_1, E_2, E_3)$,

$$\int_{E_3} \mathbf{1}_{E_1} * \mathbf{1}_{E_2} \leq \int_{E_3^\star} \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2^\star} - \mathbf{c}\,\mathrm{Distance}(\mathbf{E}, \mathcal{O}(\mathbf{E}^\star))^{\mathbf{2}}.$$

- See next slide for hypotheses and definition of distance.
- $\mathcal{O}(\mathbf{E}^\star)$ is the orbit of $\mathbf{E}^\star$ under the natural action of the affine group.
- The exponent 2 is optimal.
- Burchard's uniqueness theorem is a corollary.

Hypotheses/definition for sharpened Riesz-Sobolev

- Definition:

$$\text{Distance}(\mathbf{E}, \mathcal{O}(\mathbf{E}^\star)) \;:=\; \inf_{\psi,\mathbf{v}} \; \textstyle\sum_{j=1}^3 |E_j \,\Delta\, (\psi(E_j^\star) + v_j)|.$$

- Hypothesis: $\mathbf{E}$ is ρ–strictly admissible:

$$|E_k|^{1/d} \le (1-\rho)(|E_i|^{1/d} + |E_j|^{1/d})$$

  for all permutations (i,j,k) of $(1,2,3)$.
- $\rho > 0$.
- The constant c in the sharpened inequality depends on ρ, d.

Near-maximizers

The theorem can also be interpreted as a characterization of near-maximizers:

If

$$\int_{E_3} \mathbf{1}_{E_1} * \mathbf{1}_{E_2} \geq (1-\delta) \int_{E_3^\star} \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2^\star}$$

then

$$\text{Distance}(\mathbf{E}, \mathcal{O}(\mathbf{E}^\star)) \leq C\delta^{1/2} \max_j |E_j|$$

(with the same hypotheses).

• I originally proved this for $d = 1$ by showing that certain associated superlevel sets nearly realize equality in the 1D Brunn-Minkowski inequality.

• In Wednesday's lecture I'll explain an alternative method, which is relatively simple for $d = 1$, and has an elaboration which applies in higher dimensions.

• Today, I'll conclude by introducing two related inequalities, for which maximizers had not previously been known, but can be characterized by adaptations of this same method.

Which sets have largest Fourier transforms?

• Consider the functional

$$\frac{\|\widehat{\mathbf{1}_E}\|_q}{|E|^{1/p}}$$

of sets $E \subset \mathbb{R}^d$.

• Here p, q are conjugate exponents, and $2 < q < \infty$.

• The maximum ratio is strictly less than in the Hausdorff-Young inequality.

• The affine group is a group of symmetries.

Sets with largest Fourier transforms

Theorem. Let $d \geq 1$. Let $2m$ be an even integer ≥ 4. Let $q > 2$ be **sufficiently close to** $2m$. Then

$$E \text{ maximizes } \frac{\|\widehat{\mathbf{1}_E}\|_q}{|E|^{1/p}} \text{ if and only if } E \text{ is an ellipsoid.}$$

(The case $q = 2m$ is a corollary of Burchard's theorem.)

The case of general q remains open.

Numerical work of (the speaker and) Jon Wilkening strongly suggests that for $d = 1$, intervals are *local* maximizers for arbitrary q. We believe that this can be made rigorous (mechanically assisted) for $q = 3$.

Sets with largest Fourier transforms

- For even integer exponents q,

$$\|\widehat{f}\|_q^q = \langle f * f * \cdots * f,\, f \rangle.$$

- The techniques that I will explain for the Riesz-Sobolev inequality also apply to these higher-order multiple convolutions. They give

$$\|\widehat{\mathbf{1}_E}\|_q \le \|\widehat{\mathbf{1}_{E^\star}}\|_q - c\,\text{distance}(E, \mathcal{O}(E^\star))^2 \quad \text{if } |E| = 1.$$

- This stronger bound can be proved to be stable under small perturbations of q.
- The theorem follows.

Brascamp-Lieb-Luttinger inequality (1974)

Let $m \geq 2$ and $n \geq m$. Consider

$$\boxed{\Phi(f_1, f_2, \ldots, f_n) = \int_{\mathbb{R}^m} \prod_{j=1}^{n} (f_j \circ L_j)\, d\mathbf{x}}$$

where $L_j : \mathbb{R}^m \to \mathbb{R}^{\mathbf{1}}$ are (distinct) surjective linear mappings (with no common nullspace).

Theorem. $\boxed{\Phi(f_1, \ldots, f_n) \leq \Phi(f_1^\star, \ldots, f_n^\star).}$

The case of nonnegative functions follows directly from the fundamental case of indicator functions of arbitrary sets.

Hölder-Brascamp-Lieb/BLL functional symmetry group

• A BLL functional defined by integration over $\mathbb{R}^m$ has an m**–dimensional group of symmetries**:

$$\int_{\mathbb{R}^m} \prod_{j=1}^{n} \mathbf{1}_{E_j}(L_j(x))\, dx$$

is invariant under any translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$ of $\mathbb{R}^m$.

• Each E_j is thereby translated by $L_j(v)$, so $|E_j|$ is unchanged.

• Maximizing tuples can be unique only up to the action of this group.

• Gowers forms are interesting examples.

Example: Gowers forms and norms

- For $k \geq 2$ the Gowers **forms** are

$$\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k) = \int_{x\in\mathbb{R}} \int_{h\in\mathbb{R}^k} \prod_{\alpha\in\{0,1\}^k} f_\alpha(x + \alpha \cdot h)\, dh\, dx$$

where $f_\alpha : \mathbb{R} \to [0, \infty]$.

- There are 2^k functions f_α; integration is over $\mathbb{R}^{k+1}$.

- Gowers **norms** are

$$\|f\|_{U^k}^{2^k} = \mathcal{T}_k(f, f, \ldots, f).$$

Example: Gowers forms and norms (continued)

- $\mathcal{T}_k(f_\alpha : \alpha \in \{0,1\}^k) \le A_k \prod_\alpha \|f_\alpha\|_{L^{p_k}}$ where p_k is dictated by scaling.
- Extremizing tuples are tuples of Gaussians; Eisner-Tao found the optimal constant in the inequality.
- C. showed that sets whose indicator functions have nearly maximal Gowers norms, among sets of specified Lebesgue measure, are nearly equal to intervals.
- My student Anh Nguyen is working to prove the corresponding stability result: Functions of nearly maximal Gowers norm are nearly (in norm) equal to Gaussians.

Maximizers of the Brascamp-Lieb-Luttinger inequality?

- Burchard characterized (admissible) extremizers of BLL functionals for $n = m + 1$ (one more function than dimension). This is a corollary of her theorem on the Riesz-Sobolev inequality ($m = 2$, $n = 3$).

- [Flock–C] extended Burchard's method to establish uniqueness (up to symmetries) of maximizers for $m = 2$ for an arbitrarily large number of functions f_j, but were unable to generalize to $m > 2$.

Theorem: Maximizers of the BLL inequality

Assumptions/notation.

- $L_j : \mathbb{R}^m \to \mathbb{R}^1$ surjective linear mappings. (Natural nondegeneracy hypotheses.)
- n = number of sets E_j is $> m$.
- $(|E_j| : 1 \le j \le n)$ **strictly admissible**.

Conclusion. If $\Phi(E_1, \ldots, E_n) = \Phi(E_1^\star, \ldots, E_n^\star)$ then E_j are intervals with compatibly situated centers.

This is proved under a genericity hypothesis, which I suspect is not necessary.

Admissibility is equivalent to this simple necessary condition:
If E_j are intervals of the specified lengths centered at 0, and if $|E_j|$ is decreased for any index j, then the functional Φ decreases.

More

- The method is to directly prove the sharp quantitative version:

$$\Phi(\mathbf{E}) \leq \Phi(\mathbf{E}^{\star}) - c\,\mathrm{distance}(\mathbf{E}, \mathcal{O}(\mathbf{E}^{\star}))^2$$

where $\mathcal{O}(\mathbf{E}^{\star})$ is the orbit of $\mathbf{E}^{\star}$ under the m–dimensional translation symmetry group identified on earlier slides.

- The weaker uniqueness statement is then a corollary.

Lecture 2

Proof of the sharpened Riesz-Sobolev inequality

The Quest for Coercivity

$$\int_{E_3} \mathbf{1}_{E_1} * \mathbf{1}_{E_2} \le \int_{E_3^\star} \mathbf{1}_{E_1^\star} * \mathbf{1}_{E_2^\star} - \mathbf{c}\,\text{Distance}\,(\mathbf{E}, \mathcal{O}(\mathbf{E}^\star))^{\mathbf{2}}.$$

- $\mathbf{E} = (E_1, E_2, E_3)$
- $E_j \subset \mathbb{R}^d$
- $E_j^\star =$ ball B_j centered at 0 with $|B_j| = |E_j|$
- $\mathcal{O}(\mathbf{E}^\star) =$ orbit of $(E_1^\star, E_2^\star, E_3^\star)$ under action of the measure-preserving affine group.
- $(|E_1|, |E_2|, |E_3|)$ satisfies admissibility hypothesis.

Notation — symmetric formulation

$$\mathcal{T}(E_1, E_2, E_3) = \iint_{x_1+x_2+x_3=0} \prod_{j=1}^{3} \mathbf{1}_{E_j}(x_j)\, d\lambda(\mathbf{x})$$

with $\mathbf{x} = (x_1, x_2, x_3)$.

By replacing E_3 by $-E_3$ we may convert the problem to the study of maximizers of $\mathcal{T}$, among ordered triples $\mathbf{E}$ of sets of specified measures.

δ will systematically denote distance$(\mathbf{E}, \mathcal{O}(\mathbf{E}^\star))$.
Keep in mind the normalizations $|E_j| \asymp 1$ and $|E_j \,\Delta\, B_j| \leq O(\delta)$.

Part 0: A tool: Steiner symmetrization

- Let $d > 1$.
- Regard $\mathbb{R}^d$ as $\mathbb{R}^{d-1} \times \mathbb{R}^1$
- For $E \subset \mathbb{R}^d$ define $\mathbf{E}^\dagger \subset \mathbb{R}^d$ by: For each $x' \in \mathbb{R}^{d-1}$,

  $\{\mathbf{t} : (\mathbf{x}', \mathbf{t}) \in \mathbf{E}^\dagger\} =$ 1D symmetrization of $\{\mathbf{t} : (\mathbf{x}', \mathbf{t}) \in \mathbf{E}\}$.
- By conjugating with an arbitrary rotation of $\mathbb{R}^d$, define Steiner symmetrization in an arbitrary direction.
- Iterating Steiner symmetrizations in a(n appropriate) dense set of directions produces the radial symmetrization $E^\star$.

Steiner symmetrization and skew shifts

If B is a ball and $L : \mathbb{R}^d \to \mathbb{R}^d$ is a skew shift

$$L(x', x_d) = (x', x_d + x' \cdot v)$$

then applying Steiner symmetrization to the ellipsoid $L(B)$ reproduces the ball B.

Monotonicity and a structural feature

For arbitrary measurable subsets of $\mathbb{R}^d$,

$$\mathcal{T}(\mathbf{E}) \leq \mathcal{T}(\mathbf{E}^\dagger)$$

where $\mathbf{E}^\dagger = (E_1^\dagger, E_2^\dagger, E_3^\dagger)$.

This is simply the one-dimensional Riesz-Sobolev inequality combined with Fubini, since

$$\mathcal{T}_{\mathbb{R}^d}(\mathbf{E}) = \int_{x_1'+x_2'+x_3'=0} \mathcal{T}_{\mathbb{R}^1}(\mathbf{E_1}(\mathbf{x_1'}), \mathbf{E_2}(\mathbf{x_2'}), \mathbf{E_3}(\mathbf{x_3'}))\, d\lambda_{\mathbb{R}^{d-1}}(\mathbf{x}')$$

where

$$E_j(x') = \{t \in \mathbb{R} : (x', t) \in E_j\}.$$

Part 1: A flow of sets.

A flow on subsets of $\mathbb{R}$

There exists a flow $[0,1] \ni t \mapsto E(t)$ on **all** measurable subsets $(0 < |E| < \infty)$ of $\mathbb{R}^1$ satisfying

- $E(0)$ arbitrary; $E(1) = E(0)^\star$.
- Measure-preserving: $|E(t)| \equiv |E|$
- Continuous: $\lim_{s \to t} |E(s) \,\Delta\, E(t)| = 0$.
- Monotonic Riesz-Sobolev functional: $t \mapsto \mathcal{T}(E_1(t), E_2(t), E_3(t))$ **is nondecreasing continuous function.**

Flow on special sets (Brascamp-Lieb)

For any **finite union of intervals** $E \subset \mathbb{R}$, define a flow $t \mapsto E(t)$ for $t \in [0, 1]$, as follows:

- $E(0) = E$.
- Each constituent interval moves rigidly at a constant speed so that its center will arrive at the origin at $t = 1$
- **until the first time of collision.**
- Stop the clock, glue together any intervals that have collided. The number of intervals decreases.
- Restart the clock, with each interval moving at a constant speed chosen so that its center will arrive at the origin at $t = 1$
- until the next collision ...

Flow on general sets

- This flow can be proved to extend, by continuity, to a flow on general Lebesgue measurable sets ($|E| < \infty$).

- The extension to arbitrary sets is apparently "folklore".

Flow in higher dimensions

There exists a noncanonical extension of this flow to higher dimensions, inheriting continuity and monotonicity of the functional, using the one-dimensional flow in the same way that Steiner symmetrization uses one-dimensional symmetrization.

One chains together one-dimensional flows in sequence, in a dense sequence of directions in $\mathbb{R}^d$ ...

Aside: Smoothing property of the flow

For any set $E \subset \mathbb{R}$ (positive, finite Lebesgue measure), for any $t > 0$, $E(t)$ is (up to a null set) a countable **union of intervals**.

I do not know any useful formula for the time derivative of $\mathcal{T}(E_1(t), E_2(t), E_3(t))$.

Application of the flow

Corollary. In order to prove the sharpened Riesz-Sobolev inequality, it suffices to analyze sets that are small perturbations of balls centered at 0. □

"Small perturbation" means:

$$|E_j \,\Delta\, E_j^\star| \le \delta_0 \max_i |E_i| \qquad \forall j$$

where δ_0 may be chosen as small as desired
(but independent of $\mathbf{E} = (E_1, E_2, E_3)$).

Disappointment?

No such flow monotonicity is known for the functionals

$$\|\widehat{f}\|_q \,/\, \|f\|_p$$

or

$$\|\widehat{\mathbf{1}_E}\|_q \,/\, |E|^{1/p}$$

when $q \notin 2\mathbb{N}$.

Bennett-Bez-Carbery have proved that the Hausdorff-Young functional fails to be monotonic under the natural nonlinear heat flow, unless $q \in 2\mathbb{N}$.

(This slide was added after the lecture was delivered.)

During the lecture, a member of the audience pointed out that in 1997, Janson had given a proof of the sharp Hausdorff-Young inequality based on monotonicity of a certain quantity under a certain flow. After examination of this proof, all agreed that what was entirely different from a flow $t \mapsto f_t$. Thus while Janson's construction proves the sharp inequality, it does not permit a reduction to small perturbations of maximizers, as would a continuous flow $t \mapsto f_t$ under which the ratio $\|\widehat{f}\|_q \,/\, \|f\|_p$ was a nondecreasing function.

Proof of the sharpened Riesz-Sobolev inequality

Part 2: Perturbative expansion.

Review: Distance to orbit of $\mathbf{E}^\star$

Given $\mathbf{E} = (E_j : j \in \{1,2,3\})$ define

$$\delta = \text{Distance}(\mathbf{E}, \mathcal{O}(\mathbf{E}^\star)) = \inf_{\psi,\mathbf{v}} \max_j \left|\left(\psi(E_j) + v_j\right) \Delta B_j\right|$$

where the infimum is taken over all $\psi, \mathbf{v}$ ...

Consider orbit of $\mathbf{E} = (E_1, E_2, E_3)$ under the affine symmetry group, and choose $\psi, \mathbf{v}$ that provide an approximately optimal approximation by balls centered at 0. Then

$$\max_j |E_j \,\Delta\, E_j^\star| \leq C\, \text{Distance}(\mathbf{E}, \mathcal{O}(\mathbf{E}^\star))$$

but minimizing this distance will not be the optimal choice ...

Essential notation

- Write $\boxed{B_j = E_j^\star}$
- Express $\boxed{\mathbf{1}_{E_j} = \mathbf{1}_{B_j} + f_j.}$ Thus

$$f_j = \begin{cases} +1 \text{ on } E_j \setminus B_j \\ -1 \text{ on } B_j \setminus E_j \\ \quad 0 \text{ else.} \end{cases}$$

  - $\|f_j\|_1 = |E_j \,\Delta\, B_j| \ll |E_j|$.
  - $\int f_j = 0$.

Perturbative expansion of functional

By multilinearity, expand

$$\langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \mathbf{1}_{E_3} \rangle = \left\langle (\mathbf{1}_{B_1} + f_1) * (\mathbf{1}_{B_2} + f_2), (\mathbf{1}_{B_3} + f_3) \right\rangle$$

into sum of 8 terms.

The order of a term is the number of factors f_j that appear. The order zero term is

$$\langle \mathbf{1}_{B_1} * \mathbf{1}_{B_2}, \mathbf{1}_{B_3} \rangle.$$

Heuristically each f_j corresponds to a factor of $|E_j \, \Delta \, B_j|$, but the relationship is less direct than that.

Constrained optimization

- We are optimizing a functional subject to constraints.

- The first variation is not necessarily zero; it is nonpositive and thus helps us.

- **Second variation is typically positive. It works against us.**

- The first variation can have quadratic or linear character, or somewhere in between.

First variation terms

- Normalize henceforth so that $\max_j |E_j| = 1$.
- The first variation term is $\boxed{\sum_{n=1}^{3} \int K_n f_n}$ where

$$\boxed{K_n = \mathbf{1}_{B_i} * \mathbf{1}_{B_j}} \text{ and } \{1,2,3\} = \{i,j,n\}.$$

- The kernel K_n satisfies

$$\inf_{x \in B_n} K_n(x) = \sup_{x \notin B_n} K_n(x),$$

is radial and nonincreasing, and has strictly negative (outward) normal derivative at the boundary of B_n.

Reduction to perturbations near boundaries

Recall $K_n = \mathbf{1}_{B_i} * \mathbf{1}_{B_j}$.

- First variation terms

$$\int K_n(x)\, f_n(x)\, dx = \int \big[K_n(x) - K_n(r_n)\big]\, f_n(x)\, dx$$
$$\leq -c \int \text{distance}(x, \partial B_n)\, \mathbf{1}_{E_n \,\Delta\, B_n}(x)\, dx$$

- $\int K_n f_n$ is:
  - $\leq -c|E_n \,\Delta\, B_n|^2$ if nearly all points of $E_n \,\Delta\, B_n$ are located within distance $C|E_n \,\Delta\, B_n|$ of the boundary of B_n,
  - **but otherwise much more negative.**

Reduction to perturbations near the boundaries

- The second and third order remainder terms in this expansion are

$$O(\max_j |E_j \, \Delta \, B_j|^2).$$

- A simple analysis exploiting this allows reduction to the case in which
every point of $E_j \, \Delta \, B_j$ lies within distance $C \max_i |E_i \, \Delta \, B_i|$ of the boundary of the ball B_j, for each index j.

Part 3: Reduction to the boundaries

Onto the boundaries

Define $\mathbf{1}_{E_j \setminus B_j} = f_j^+$ and $\mathbf{1}_{B_j \setminus E_j} = f_j^-$, so that $f_j = f_j^+ - f_j^-$.
Using polar coordinates $x = r\theta$ define $F_j^\pm \in L^2(S^{d-1})$ by

$$\mathbf{F_j^\pm}(\theta) = \int_{\mathbb{R}^+} \mathbf{f_j^\pm}(\mathbf{t}\theta)\, \mathbf{t^{d-1}}\, \mathbf{dt}$$

where $\theta \in S^{d-1}$ and $F_j \in L^2(S^{d-1})$ by $F_j = F_j^+ - F_j^-$.
Then

$$|E_j \,\Delta\, B_j|^2 \asymp \|F_j^+\|^2_{L^2(S^{d-1})} + \|F_j^-\|^2_{L^2(S^{d-1})}.$$

So our goal is:

$$\mathcal{T}(\mathbf{E}) \leq \mathcal{T}(\mathbf{E}^\star) - c\sum_j \|F_j^+\|^2_{L^2(S^{d-1})} - c\sum_j \|F_j^-\|^2_{L^2(S^{d-1})}.$$

Reformulation on $L^2(S^{d-1}) \times L^2(S^{d-1}) \times L^2(S^{d-1})$

$$
\begin{aligned}
\mathcal{T}(\mathbf{E}) \leq \mathcal{T}(\mathbf{E}^\star) &- \tfrac{1}{2}\sum_{k=1}^{3} \gamma_{\mathbf{k}} \sum_{\pm} \|F_k^{\pm}\|^2_{L^2(S^{d-1})} \\
&+ \mathcal{Q}(\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}) \\
&+ O(\delta^3)
\end{aligned}
$$

where

$$\gamma_k = -r_k^{1-d}(x \cdot \nabla)K_k(x)\big|_{|x|=r_k}, \quad (r_k \text{ is the radius of } B_k)$$

$$\mathcal{Q}(F_1, F_2, F_3) = \mathcal{Q}_1(F_2, F_3) + \mathcal{Q}_2(F_3, F_1) + \mathcal{Q}_3(F_1, F_2)$$

$$\mathcal{Q}_{\mathbf{k}}(\mathbf{F_i}, \mathbf{F_j}) = \iint_{\mathbf{S^{d-1}} \times \mathbf{S^{d-1}}} \mathbf{F_i}(\mathbf{x})\, \mathbf{F_j}(\mathbf{y})\, \mathbf{1}_{|\mathbf{r_i x} + \mathbf{r_j y}| \leq \mathbf{r_k}}\, \mathbf{d}\sigma(\mathbf{x})\, \mathbf{d}\sigma(\mathbf{y}).$$

• The idea underlying the reduction to the Cartesian product of the boundaries of the balls B_j is to exploit the assumption that f_i, f_j are supported near the boundaries, together with the continuity of $\mathbf{1}_{K_k}$.

• It turns out that even though $\mathbf{1}_{K_k}$ has jump discontinuities at the boundary of B_k, it is sufficiently continuous to permit such a reduction modulo an error that is $o(\delta^2)$.

• This step is easy, and completely elementary.

Cap operator

Quadratic forms:

$$\mathcal{Q}_k(F_i, F_j) = \iint_{S^{d-1}\times S^{d-1}} F_i(x)\, F_j(y)\, \mathbf{1}_{|r_i x + r_j y| \le r_k}\, d\sigma(x)\, d\sigma(y).$$

Each $\mathcal{Q}_k$ is "spherical convolution" with the indicator function of a $\boxed{\text{cap}}$

$$\{(x', x_d) : x_d \ge 1 - \rho\}$$

where ρ is a function of (r_1, r_2, r_3).

Diagonalization

- The linear compact self-adjoint operators on $L^2(S^{d-1})$

$$T_k(F) = \int_{S^{d-1}} F(y)\,\mathbf{1}_{|r_i x + r_j y| \le r_k}\, d\sigma(y)$$

are diagonalized by **spherical harmonics Y_n of degrees n**.

- Their eigenvalues tend to zero as $n \to \infty$.
- It suffices to show that the maximum, over n, of the optimal constants $\mathbf{A_n}$ in the inequalities

$$\mathcal{Q}(F_1, F_2, F_3) \le \mathbf{A_n} \sum_{k=1}^{3} \gamma_k \|F_k\|^2_{L^2(S^{d-1})}$$

for spherical harmonics of degree n is **strictly less than $\frac{1}{2}$.**

- $\mathcal{Q}$ and γ_k are functions of (r_1, r_2, r_3), where r_j is the radius of B_j.
- $n = 0$ can be disregarded; $\pi_0(F_j) = 0$ since $\int_{S^{d-1}} F_j = \int_{\mathbb{R}^d} f_j = 0$.

Failure?

- ☹ For both $n = 1$ and $n = 2$, $\boxed{A_n = \frac{1}{2}}$ exactly, and the proof collapses

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- ☺ Failure is good;

Failure?

- ☹ For both $n = 1$ and $n = 2$, $\boxed{A_n = \frac{1}{2}}$ exactly,
  and the proof collapses
  as was obvious from the outset without calculation.

- ☺ Failure is good; otherwise we would have proved the false conclusion that balls centered at 0 are the only maximizing triples. Ellipsoids are just as good!

$\mathbb{R}^1$ case

The case $d = 1$ is far simpler than $d > 1$, because there are no spherical harmonics of degrees > 1 for $d = 1$. We'll discuss the more typical case $d > 1$ first, then return to $d = 1$.

Part 4: Balancing

• We are analyzing the functional $\mathcal{T}(\mathbf{E}) = \langle \mathbf{1}_{E_1} * \mathbf{1}_{E_2}, \, \mathbf{1}_{-E_3} \rangle$ by Taylor expansion about $\mathbf{E}^\star$.

• Taylor expansion about some other ordered triple of ellipsoids in the orbit of $\mathbf{E}^\star$ might be better.

• Which triple provides an accurate Taylor approximation $\boxed{\textbf{?}}$

Balancing

- $\delta = \text{Distance}(\mathbf{E}, \mathcal{O}(\mathbf{E}^\star))$
- Each $E_j \,\Delta\, B_j$ is contained in a $C\delta$–neighborhood of the boundary of B_j.
- $\pi_{\mathbf{n}}$ = orthogonal projection of $L^2(S^{d-1})$ onto spherical harmonics of degree n.

Lemma. Let $d \geq 2$. There exist $\psi, \mathbf{v}$ as above such that the functions $\tilde{F}_j$ associated to the sets $\tilde{E}_j = \psi(E_j) + v_j$ satisfy

$$\pi_{\mathbf{1}}(\tilde{\mathbf{F}}_{\mathbf{1}}) = \pi_{\mathbf{1}}(\tilde{\mathbf{F}}_{\mathbf{2}}) = \mathbf{0} \qquad \text{and} \qquad \pi_{\mathbf{2}}(\tilde{\mathbf{F}}_{\mathbf{1}}) = \mathbf{0}$$

and still

$$\text{Distance}(\tilde{\mathbf{E}}, \mathcal{O}(\mathbf{E}^\star)) \leq C\delta.$$

□

Immediate benefits of spherical harmonic reduction

- $\pi_1(F_1) = \pi_1(F_2) = 0$ kills the contribution of $n = 1$ **and therefore finishes off the case $d = 1$!**.
- $\pi_2(F_1) = 0$ tempers the $n = 2$ contribution, though it doesn't quite kill it.
- Still need to prove: for each $n \geq 3$, for spherical harmonics G_j of degree n,

$$\mathcal{Q}(G_1, G_2, G_3) \leq \left(\tfrac{1}{2} - \eta\right) \sum_{k=1}^{3} \gamma_k \|G_k\|^2_{L^2(S^{d-1})}.$$

- Because the eigenvalues of our compact self-adjoint operators tend to zero, this holds for all sufficiently large n; **no uniformity need be shown.**

Hopes dashed

- This boils down mainly to the calculation of the action of spherical “convolution” with the indicator function of a spherical cap, of arbitrary radius, on $L^2(S^{d-1})$.
- One might hope to use the detailed theory of spherical harmonics to perform this calculation

Hopes dashed

- This boils down mainly to the calculation of the action of spherical “convolution” with the indicator function of a spherical cap, of arbitrary radius, on $L^2(S^{d-1})$.
- One might hope to use the detailed theory of spherical harmonics to perform this calculation
- but this hope for $d > 2$ is dashed by the realization that what is required is tantamount to explicit formulas for the indefinite integrals of all zonal harmonics (to be followed by possibly intricate exploitation of such formulas).
- The final step exploits a more conceptual approach that reduces matters from analysis, to algebra. This more holistic approach **reunites the first and second order terms** of the expansion.

Part 5: Conclusion

- The reduction to $L^2(S^{d-1})$ is **reversible** for star-shaped sets.
- Given an arbitrary ordered triple $\mathbf{G}$ of spherical harmonics, certain triples $\mathbf{E}$ are introduced, for which an appropriate bound for $\mathcal{T}(\mathbf{E})$ is equivalent to the desired bound for $\mathbf{G}$.
- Steiner symmetrization is used to directly analyze $\mathcal{T}(\mathbf{E})$.

Analysis of special sets E_j

- Fix a degree n. Let $\mathbf{G} = (G_1, G_2, G_3)$ be a nonzero ordered triple of spherical harmonics of degree n.
- Consider (in polar coordinates) the special sets

$$\mathbf{E_j} = \mathbf{E_j}(\mathbf{s}) = \{(\mathbf{t}, \theta) : \mathbf{t} \leq \mathbf{r_j} + \mathbf{s}\mathbf{G_j}(\theta)\}$$

where $s \in \mathbb{R}$ is a small parameter.
- Need only prove infinitesimal result as $s \to 0$:

$$\mathcal{T}(\mathbf{E}(s)) \leq \mathcal{T}(\mathbf{E}^{\star}) - cs^2 + o(s^2).$$

- Since $\mathcal{T}(\mathbf{E}^{\dagger}(s)) \leq \mathcal{T}(\mathbf{E}^{\star})$, it is enough to have

$$\mathcal{T}(\mathbf{E}(s)) \leq \mathcal{T}(\mathbf{E}^{\dagger}(s)) - cs^2 + o(s^2).$$

Comparison

Notation:

$$\Sigma = \{\mathbf{x}' \in (\mathbb{R}^{d-1})^3 : x_1' + x_2' + x_3' = 0\}$$
$$E_j(x') = \{t \in \mathbb{R}^1 : (x', t) \in E_j\}$$
$$\mathcal{T}_1 = \text{ trilinear Riesz-Sobolev form for } \mathbb{R}^1.$$

Then

$$\mathcal{T}(\mathbf{E}) = \int_\Sigma \mathcal{T}_1(E_1(x_1'), E_2(x_2'), E_3(x_3'))\, d\lambda(\mathbf{x}')$$
$$\mathcal{T}(\mathbf{E}^\dagger) = \int_\Sigma \mathcal{T}_1(E_1(x_1')^\star, E_2(x_2')^\star, E_3(x_3')^\star)\, d\lambda(\mathbf{x}').$$

Riesz-Sobolev inequality for $\mathbb{R}^1$ says second integrand majorizes first pointwise.

Effect of Steiner symmetrization on functional

For $E_j(s) = \{(t,\theta) : t \le r_j + sG_j(\theta)\}$:

- Vertical slices of these sets are one-dimensional intervals.
- Let $c_j(x')$ be the center of the slice of E_j above the point $x' \in \mathbb{R}^{d-1}$.
- The functional increases unless the centers satisfy

  $$c_1(x_1') + c_2(x_2') + c_3(x_3') = 0 \quad \text{whenever} \quad x_1' + x_2' + x_3' = 0;$$

  the gain is proportional to $|c_1(x_1') + c_2(x_2') + c_3(x_3')|^2$.
- Infinitesimal version: There is a gain of magnitude s^2 unless the three spherical harmonics G_j satisfy a related algebraic equation.

Gain from Steiner symmetrization

With the above hypotheses and notations,

$$\mathcal{T}(\mathbf{E}_{\mathbf{G}}(s)) \leq \mathcal{T}(\mathbf{E}^{\dagger}) - cs^2 \|P^{\sharp}_{\mathbf{G}}\|^2 + O(s^3)$$

where $P^{\sharp}_{\mathbf{G}}$ is a polynomial defined in terms of **G** on the next slide.

The next few slides are essentially high school algebra. I have not succeeded in explaining this step in a conceptual way, and am secretly hoping to run out of time so as to have an excuse for skipping it.

Nature of the gain

Regard x_d as a function of x' via $x_d = (r_j^2 - |x'|^2)^{1/2}$.
Expand $G_j(x', x_d)$ as a sum of monomials in $x = (x', x_d)$.
Define $G_j^{\mathrm{odd}}(x', x_d) =$ sum of monomials having odd degrees with respect to x_d.

$$P_j(x') = r_j^{2-d-n} x_d^{-1} G_j^{\mathrm{odd}}(x', x_d)$$
$$P_{\mathbf{G}}^{\sharp}(\mathbf{x}') = \sum_{j=1}^{3} P_j(x_j').$$

• Since we gain $cs^2\|P^\sharp_{\mathbf{G}}\|^2$, it would suffice to have $\mathbf{G} \neq 0 \implies P^\sharp_{\mathbf{G}} \neq 0$ for spherical harmonics of arbitary degrees ≥ 3.

• Alas: For any n, there do exist nonzero ordered triples of spherical harmonics $\mathbf{G}$ that satisfy $P^\sharp_{\mathbf{G}} \equiv 0$. ☹

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• However, we are free to **apply Steiner symmetrization in an arbitrary direction.** It suffices to show that for $n \geq 3$, if $\boxed{\text{all}}$ rotations of $\mathbf{G}$ satisfy $P^\sharp_{\mathbf{G}} \equiv 0$, then $\mathbf{G} \equiv 0$.

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• However, we are free to **apply Steiner symmetrization in an arbitrary direction.** It suffices to show that for $n \geq 3$, if all rotations of $\mathbf{G}$ satisfy $P^\sharp_{\mathbf{G}} \equiv 0$, then $\mathbf{G} \equiv 0$.

• This is true.

The same kind of algebraic analysis establishes the required inequality for $n = 2$.

(Recall that for degree $n = 2$, balancing reduces matters to the case in which $G_1 \equiv 0$, but G_2, G_3 remain arbitrary spherical harmonics of degree 2.)