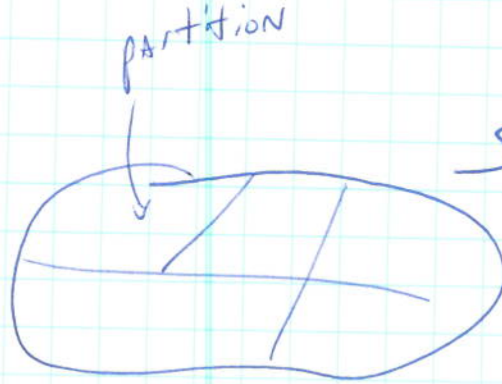
Decoupling

Larry,
Talk I
Jan. 23, 2017
①

$\Omega \subseteq \mathbb{R}^n$

$\operatorname{Supp} \hat{F} \subseteq \Omega$



$f = \sum_\theta f_\theta$

$f_\theta = \int \hat{f}(\omega) e^{2\pi i \omega x} d\omega$ $\qquad \Omega = \bigsqcup \theta$

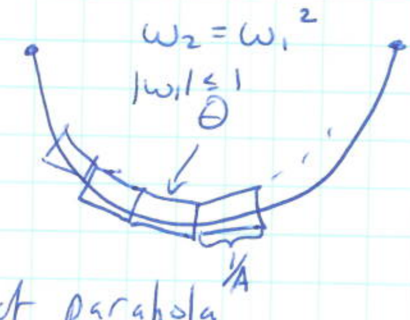
$\|f\|_2 = \left(\sum_\theta \|f_\theta\|_2^2 \right)^{1/2}$

$D_p(\underset{\substack{\uparrow \\ \text{region}}}{\Omega} = \bigsqcup \underset{\substack{\uparrow \\ \text{pieces}}}{\theta}) =$ Best constant s.t. $\|f\|_p \le D_p \left(\sum_\theta \|f_\theta\|_p^2 \right)^{1/2}$

Parabola $(n=2)$ $\qquad$ example

$\#\{\theta\} = A$

Each $\theta \approx \frac{1}{A^2} \times \frac{1}{A}$ rectangles



$\Omega = \bigsqcup \theta = \frac{1}{A^2}$ - neighborhood of parabola

Thrm. (BD) If $2 \le p \le 6$, then $D_p(A) \lessapprox A^\varepsilon$

Variations:

Paraboloid
Cone
Sphere } BD

Moment curve } (BDG)

If f_θ indep. random variables, $f = \sum f_\theta$

$|f(x)| \sim \left(\sum |f_\theta|^2\right)^{1/2}$ w/ high probability

$\Rightarrow \|f\|_p \sim \|$ " $\|_p \quad 1 < p < \infty$

$\Rightarrow \|f\|_p \lesssim \left(\sum_\theta \|f_\theta\|_p^2\right)^{1/2}$

Examples If supp f_θ disjoint, $\|f\|_p = \left(\sum \|f_\theta\|_p^p\right)^{1/2} \overset{?}{\gtrsim}$

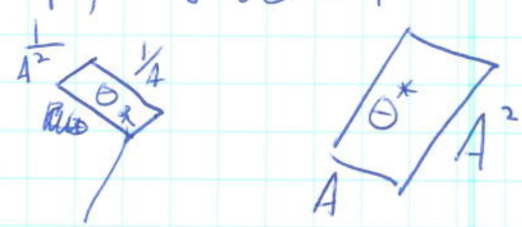
If $|f_\theta| \approx \chi_E \quad \forall \theta$, the best we can hope for is

$|f| \approx A^{1/2} \chi_E$.

recall $\#\{\theta\} = A$

η_θ smooth bump on θ, $\int \eta_\theta = 1$

$f_\theta = \eta_\theta^\vee$



$|f| \sim \chi_{\theta^*}$

pick a pt. ω_θ.

$f_\theta(x) \approx e^{2\pi i \omega_\theta x} \chi_{\theta^*}$ "oscillates"

$f = \sum f_\theta$

$f(0) = A$

$|f(x)| \sim A$ for $|x| = 1$

$\|f\|_p \sim A$ for $p > 4$ (exercise)

$\|f_\theta\|_p \sim |\theta^*|^{1/p} = A^{3/p}$

$D_p(A) \le A^\varepsilon \Rightarrow A \lesssim A^\varepsilon \cdot A^{1/2} \cdot A^{3/p} \Rightarrow p \le 6$

(recall $\|f\|_p \le D_p \left(\sum_\theta \|f_\theta\|_p^2\right)^{1/2}$

Guth ③

More examples:

Translation $\quad f_\theta(x) = \eta_\theta^\vee(x - v)$

Linear transform $\quad f_\theta(x) = \sum_j c_j \underbrace{\eta_\theta^\vee(x - v_j)}_{\text{wave packet}}$

$\forall \theta$, $h_\theta = A$ wave packets

$$h_\theta = \sum_{j=1}^{A} c_{j,\theta}\, \eta_\theta^\vee(x - v_j)$$

$\|h_\theta\|_{L^6} \sim A^{2/3} \qquad |c_{j,\theta}| = 1$

Dec. $\Rightarrow \|h\|_6 \lesssim A^{\frac{1}{2}+\frac{2}{3}+\varepsilon} = A^{\frac{7}{6}+\varepsilon}$

$C = \{x : |h(x)| \sim A\} \qquad |C| \lesssim A^{1+\varepsilon}$

$S = A$ unit balls

$|\text{w.p.} \cap S| \sim 1$, can chose $c_{j,\theta}$ ...

Application to exponential sums:

"use this for periodicity in proof"

Prop. If $\Omega = \bigsqcup \theta$ (open), $\omega_\theta \in \theta \cap \mathbb{Z}^n$

$g = \sum_\theta c_\theta e^{2\pi i \omega_\theta x}$. Then $\|g\|_{L^p([0,1]^n)} \lesssim D_p \cdot \left(\sum c_\theta^2\right)^{1/2}$

$c_\theta = \|g_\theta\|_{L^p([0,1]^n)}$

"note what makes this proposition special is $\mathbb{R}^n \rightarrow [0,1]^n$"

Sketch proof: $R \qquad B(\omega_\theta, \frac{1}{R}) \subseteq \theta$

$\eta \qquad \operatorname{supp} \hat{\eta} \subseteq B(1/R)$

$\eta \sim 1$ on B_R, rap. decay

$f_\theta = \eta g_\theta \qquad \|f_\theta\|_p \sim R^{n/p} |c_\theta|$

$f = \eta g \qquad \|f\|_p \sim R^{n/p} \|g\|_p$

Thm of BDG on the Moment Curve is connected to no. theory.

Conn. to # theory

$2s = n(n+1)$

$$\int |E|^{2s} = \int_{[0,1]^n} E^s \cdot \bar{E}^s = \int \sum_{\substack{a_1,\dots,a_s=0\\ b_1,\dots,b_s=0}}^{A} e((a_1+\dots+a_s-b_1-\dots-b_s)x_1) \cdot e((a_1^2+\dots-b_1^2-\dots)x_2) \cdots$$

$$= \# \{ 0 \le a_j, b_j \le A,\ a_j, b_j \in \mathbb{Z} : a_1^j + \dots + a_s^j = b_1^j + \dots + b_s^j,\ j=1,\dots,n \}$$

"diophantane solutions"
"important in no. theory"
" $n=3$ trevor Wooly "

A hard problem in no. theory.

Proof next lecture.
Uses orthogonality
geometry
Scales. ← talk II

A view from above - he has worked a lot to explain decoupling.

Today's focus: proof of decoupling

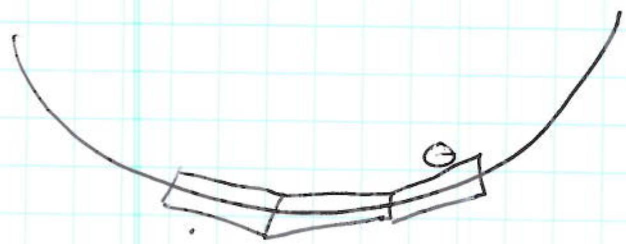
study guide for the ℓ^2 decoupling theorem (BD)

$$\Omega = \bigsqcup \theta$$

D_p = best constant s.t. $\forall F$ w/ supp $\hat{F} \subseteq \Omega$

$$\|f\|_p \leq D_p \left(\sum_\theta \|f_\theta\|_p^2 \right)^{1/2}$$

Parabola



$\#\{\theta\} = A$

each $\theta \approx \frac{1}{A^2} \times \frac{1}{A}$

Thrm $2 \leq p \leq 6$ $\quad D_p(A) \lessapprox A^\varepsilon$

Lem 1 $L : \mathbb{R}^n \circlearrowright$ linear affine c.o.v.

$$D_p(L\Omega = \bigsqcup L\theta) = D_p(\Omega = \bigsqcup \theta)$$

Pf// $g_{L\Omega} = \sum g_{L\theta}$ $\quad \|g\|_p = |J_{ac}|^{1/p} \| \quad \|$

$\|g_{L\theta}\|_p = |J_{ac}|^{1/p} \|f_\theta\|_p$

Lem 2 If $A = A_1 \cdot A_2 \to D_p(A) \lessapprox D_p(A_1) D_p(A_2)$ //

pf// $\Omega = \bigsqcup \tau$ $\quad \#\tau = A_2$

$$\|f\|_p^2 \leq (D_p(A_2))^2 \sum_\tau \|f_\tau\|_p^2$$

by defn.

break Ω into τs then into θs //

$$\leq (D_p(A_1))^2 (D_p(A_2))^2 \sum_\theta \|f_\theta\|_p^2$$

* I think he uses F & f to mean same thing — notetaker.

Joke proof (WRONG but illuminating)

Induction on A. $D_p(A) \lesssim A_p(A^{1/2}) \cdot D_p(A^{1/2})$

$\leq A^{\varepsilon/2} \cdot A^{\varepsilon/2} = A^{\varepsilon}$.

"Symbol of joke proof"

Problem: missing constant

bigger problem: $\underline{\underline{\text{no}}}$ base case.

Rmk 1: $D_6(10^{10,000}) \leq 10^{500} = (10^{10,000})^{1/2}$

Lem. 2 iterate $\rightarrow$ $D_6(A) \lesssim A^{1/10}$ $\forall A$.

Rmk 2: If $D_p(A) \lesssim D_p(A^{1/2})\, D_p(A^{1/2 - \varepsilon})$

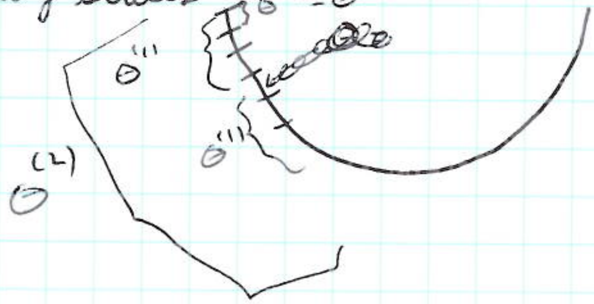
$\Rightarrow D_p(A) \lesssim A^{\varepsilon}$

"then situation is good enough to close induction"

If $D_p(A) \lesssim D_p(A^{1/2})\, D_p(A^{1/4}) \cdot A^{\varepsilon/8} \cdot D_p(A^{1/8})$

$\Rightarrow D_p(A) \lesssim A^{\varepsilon}$.

Many scales



base scale Θ

Collection of scales (to make bigger scales)

$\Theta^{(1)}, \Theta^{(2)}, \ldots$

made up of collection of

$$\#\{\Theta^{(j)} \subseteq \Theta^{(j+1)}\} = \frac{A_j}{A_{j+1}}$$

6th Talk II ③

$$E_j(f) = \left(\sum_{\theta^{(j)}} \|f_{\theta^{(j)}}\|_p^2\right)^{1/2}$$

W.T.S. $E_S(f) \lesssim A^{\varepsilon} \cdot E_0(f)$

"Errase joke proof & look at:"

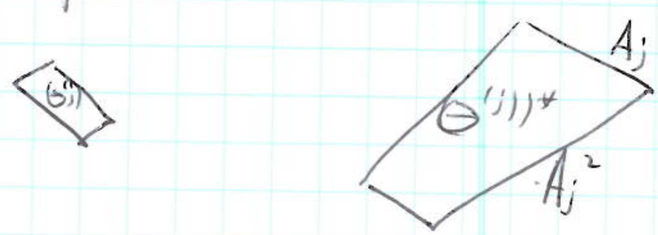
$$E_{j+1}(f) \lesssim D_p\left(\frac{A_j}{A_{j+1}}\right)$$

Suppose $\forall f, \exists j$ s.t. $E_{j+1}(f) \lesssim \left(\frac{A_j}{A_{j+1}}\right)^{\varepsilon} E_j(f)$ $(**)$

$(**) \implies D_p(A) \lesssim A^{\varepsilon}$.

Each $F_{\theta^{(j)}}$ has wave packet decomp.

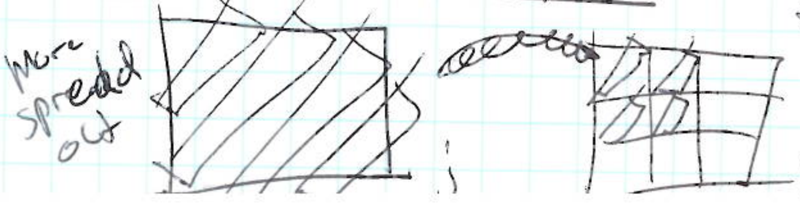
$|F_{\theta^{(j)}}| \approx$ constant on translates of $(\theta^{(j)})^*$



<u>Orthogonality</u> $F_{\theta^{(j)}}$ are (morally) orthog. on $\square_{j+1}$

$$\|f_{\theta^{j+1}}\|^2_{L^2(\square_{j+1})} \sim \sum \|f_{\theta_j}\|^2_{L^2(\square_{j+1})}$$

<u>Hypothetical bad example</u> lots of wave packets jammed together



more concentrated so the L^p norms are very

We will classify this as a "bad" scale. (4) Guth TALK II

Let's quantify "good" & "bad" scales.

In each $\square_j$, $f_{\theta^{(j)}} \approx W_j$ wave packets

$W_j = A_j$ (max). $\quad W_j = 1$

$1 \le W_j \le A_j$ or $W_j = A_j^{\Theta_j}$ $\quad 0 \le \Theta_j \le 1$

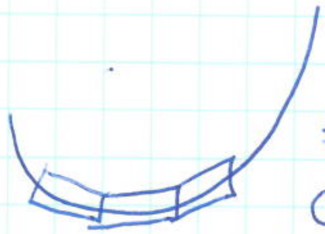
<u>defn:</u> j is ε-good if $\Theta_{j+1} \ge \Theta_j - \varepsilon$.

Lemma: $\exists$ $j \in \{0, \dots, \frac{1}{\varepsilon}\}$ which is ε-good.

Prove by contradiction.

We will look at a good scale, use orthogonality, and compute $\longleftarrow$ Next time.

Larry Guth
Talk III
(I)



$\#\theta = A$

$\theta = \frac{1}{A^2} \times \frac{1}{A}$ rectangle

$$\boxed{\begin{array}{l} \|f\|_p \le D_p(A) \left(\sum_\theta \|f_\theta\|_p^2 \right)^{1/2} \quad (D) \\ \underline{\text{THM}} \quad 2 \le p \le 6, \quad D_p(A) \lessapprox A^\varepsilon. \end{array}}$$

option 1

Sketch proof
- Many Scales
- Orthogonality
- Geometry

"only 3 ingredients"

local decoupling -

Prop Suppose box is $\square$ } A^2. $\eta_\square \sim 1$ on $\square$, rap. dec.

$$\|f\|_{L^p(\square)} \le 10 \cdot D_p(A) \left(\sum_\theta \| \quad \|_{L^p}^2 \right)^{1/2}$$

Can I write f on $\square$?
no, but I can put $\eta_\square f_\theta$

Take $\operatorname{supp} \widehat{\eta_\square} \subseteq B(\frac{1}{A^2})$.
Then $\operatorname{supp}(\widehat{\eta_\square f_\theta}) \subseteq \frac{1}{A^2}$ - neighb. of θ. $\approx \theta$.
"morally"

"Let's just write this... instead of not super rigorous..."

Apply Decoupling to $\eta_\square f$

$$\|f\|_p = \left(\sum_\square \|f\|_{L^p(\square)}^p \right)^{1/p} \lessapprox D_p(A) \cdot \left(\sum_\square \left[\sum_\theta \|f_\theta\|_{L^p(\square)}^2 \right]^{\frac{p}{2}} \right)^{1/p}$$

option 2

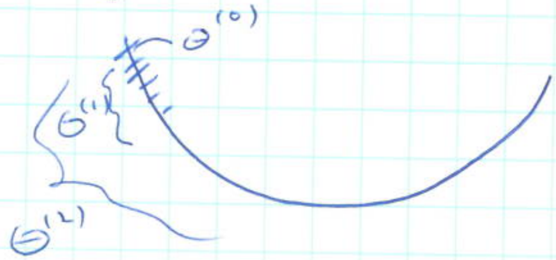
We have two upper bounds. Which is better?
It Depends on the function.

<u>example</u>: $\|f_\theta\|_p = 1 \ \forall$ $\operatorname{supp} f_\theta \subseteq$ different $\square$.

(1) $A^{1/2}$
(2) $A^{1/p}$

<u>exercise:</u> ② $\le$ ①   hint: (Minkowski inequality) ②

Many scales:



$\#\{\theta^{(j)}\} = A_j$ "# of scales"

$A_0 = A,\ A_1 = A^{1/2}, \dots, A_j = A^{\frac{1}{2^j}}, \dots \to A_S = 1$

$f_{\theta^{(j)}} = \sum_{A_j \times A_j^2}$ wave packets

"behaves nicely as $j \nearrow$"

$\square_j :=$ side A_j^2

Study $E_j(f) := \left(\sum_{\square_j \subseteq \mathbb{R}^d} \left(\sum_{\theta^{(j)}} \|f_{\theta^{(j)}}\|^2_{L^p(\square_j)} \right)^{p/2} \right)^{1/p}$ NEW ②

outline proof:

$$\|f\|_{L^p} = E_S(f) \lesssim A^{\varepsilon} E_0(f) \le A^{\varepsilon} \left(\sum_\theta \|f_\theta\|_p^2 \right)^{1/2}.$$

our exercise above.

① Find good j: $E_{j+1}(f) \lesssim \left(\frac{A_j}{A_{j+1}}\right)^{\varepsilon} E_j(f)$.

② Induction on scales (Lect. 2)

Portrait of $\{f_{\theta^{(j)}}\}$

"Key features in picture"

Important boxes: $\square_j$, side A_j, $X_j = \bigcup \square_j$

$\forall\, \square_j \subseteq X_j$

D_j impo- $f_{\theta^{(j)}}$

Each ω_j wave p. of height H_j.

Here, $D_j = \omega_j = 2$ Also two directions

3

Guth Talk III

$E_j(f)$ vs $\|f\|_{L^2(X_j)}$

$$E_j(f) := \underbrace{(\#\text{ boxes } X_j)^{\frac{1}{p}} D_j^{1/2} H_j \left(\frac{w_j}{A_j}|\Box_j|\right)^{\frac{1}{p}}}$$

"volume X_j"

$$= \left(|X_j|\,\frac{w_j}{A_j}\right)^{1/p} D_j^{1/2} H_j$$

$\|f\|_{L^2(X_j)}$ = same w/ $p=2$

$$\frac{E_{j+1}}{E_j} = \frac{\|f\|_{L^2(X_{j+1})}}{\|f\|_{L^2(X_j)}} \underbrace{\left(\frac{|X_j|}{|X_{j+1}|}\right)^{\frac{1}{2}-\frac{1}{p}} \left(\frac{w_j}{w_{j+1}A_j^{1/2}}\right)^{\frac{1}{2}-1/p}}$$

"extra correction factor"

handle w/ "orthogonality" "geometry"

Recall: j is ε-good if $w_{j+1} \geq w_j^{\frac{1-\varepsilon}{2}}$

"want RHS small so don't want w_j to big compared w_{j+1}"

(Special case: $X_{j+1} = X_j$)

If T is a wave packet, $\lambda(T) := \dfrac{|T \cap X_{j+1}|}{|T|}$

Dyadic pig. $\lambda(T) \approx \lambda$, $\forall\, T$.

<u>Lem.</u> $\|f\|^2_{L^2(X_{j+1})} = \lambda\, \|f\|^2_{L^2(X_j)}$

Pf// $LHS = \sum_{\Box_{j+1} \subseteq X_{j+1}} \|f_{\Theta_j}\|^2_{L^2(\Box_{j+1})}$ $\cancel{= \sum_{\Theta_j} \|f\|^2_{L^2(X)}}$

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Guth Talk III

$$= \sum_{\theta^{(j)}} \|f_{\theta^{(j)}}\|^2_{L^2(X_{j+1})}$$

"Swap sums"

$$= \sum_{\theta^{(j)}} \lambda \|f_{\theta^{(j)}}\|_{L^2(X_j)} \underset{\text{orthogonality}}{=} \lambda \|f\|_{L^2(X_j)}$$

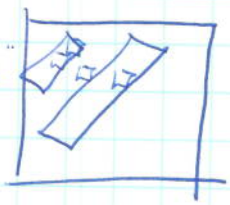
orthogonality.

$\square$

$$\frac{E_{j+1}}{E_j} \lessapprox \lambda^{\frac{1}{2}} \rho^{\frac{1}{p} - \frac{1}{2}} \quad (w\text{-term}) \qquad \rho = \frac{|\Box_j \cap X_{j+1}|}{|\Box_j|}$$

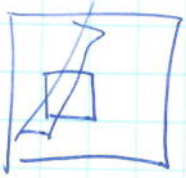
Question: How are λ, ρ related?

Triv. $\left(\frac{w_j}{A_j}\right) \lambda \lesssim \rho$ pf// by pic. "use definitions"



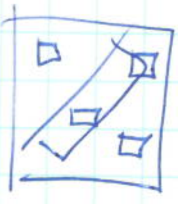
Broad

Ex. 1



small box in middle & a wave packet.
Here, $\lambda \approx \rho^{1/2}$

Ex 2 Spread out little middle box. & 1 wave packet.
Here, $\lambda \lessapprox \rho^{1/2}$



Narrow

Ex. 3

little boxes in column. Wave packet ~~parallel~~ parallel.
$\lambda \sim 1 \gg \rho^{1/2}$.

He uses these 3 examples to "test" what is happening.

check these cases to grasp $p=6$